WENTWORTH-SMITH MATHEMATICAL SERIES

PLANE AND SPHERICAL TRIGONOMETRY

BY

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AND

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PREFACE

In preparing a work to replace the Wentworth Trigonometry, which has dominated the teaching of the subject in America for a whole generation, some words of explanation are necessary as to the desirability of the changes that have been made. Although the great truths of mathematics are permanent, educational policy changes from generation to generation, and the time has now arrived when some rearrangement of matter is necessary to meet the legitimate demands of the schools.

The principal changes from the general plan of the standard texts in use in America relate to the sequence of material and to the number and nature of the practical applications. With respect to sequence the rule has been followed that the practical use of every new feature should be clearly set forth before the abstract theory is developed. For example, it will be noticed that the definite uses of each of the natural functions are given as soon as possible, that the need for logarithmic computation follows, that thereafter the secant and cosecant assume a minor place, and that a wide range of practical applications of the right triangle awakens an early interest in the subject. The study of the functions of larger angles, and of the sum and difference of two angles, now becomes necessary to further progress in trigonometry, after which the oblique triangle is considered, together with a large number of practical, nontechnical applications.

The decimal division of the degree is explained and is used enough to show its value, but it is recognized that this topic has, as yet, only a subordinate place. It seems probable that the decimal fraction will in due time supplant the sexagesimal here as it has in other fields of science, and hence the student should be familiar with its advantages.

Such topics as the radian, graphs of the various functions, the applications of trigonometry to higher algebra, and the theory of trigonometric equations properly find place at the end of the course in plane trigonometry. They are important, but their value is best appreciated after a good course in the practical uses of the subject.

They may be considered briefly or at length as the circumstances may warrant.

In the spherical trigonometry the same principles have been followed, the practical preceding the theoretical, and the number of applications being increased, but the technical work on astronomy is relegated to textbooks on that subject.

The authors have sought to give teachers and students all the material needed for a thorough study of plane and spherical trigonometry, with more problems than any one class will use, thus offering opportunity for a new selection of examples from year to year, and allowing for the omission of the more theoretical portions of Chapters IX–XII of the Plane Trigonometry if desired.

The tables have been arranged with great care, every practical device having been adopted to save eye strain, all tabular material being furnished that the student will need, and an opportunity being afforded to use angles divided either sexagesimally or decimally, as the occasion demands.

The answers have been placed at the back of the book, experience having shown that, in trigonometry as well as in other subjects, this is better than to incorporate them in the text.

It is hoped that the care that has been taken to arrange all matter in the order of difficulty and of actual need, to place the practical before the theoretical, to eliminate all that is not necessary to a clear understanding of the subject, and to present a page that is at the same time pleasing to the eye and inviting to the mind will commend itself to and will meet with the approval of the many friends of the series of which this work is a part.

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Nov. 26, 1918

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PLANE TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

1. The Nature of Arithmetic. In arithmetic we study computation, the working with numbers. We may have a formula expressed in algebraic symbols, such as $a = bh$, but the actual computation involved in applying such a formula to a particular case is part of arithmetic.

Arithmetic enters into all subsequent branches of mathematics. It plays such an important part in trigonometry that it becomes necessary to introduce another method of computation, the method which makes use of logarithms.

2. The Nature of Algebra. In algebra we generalize arithmetic. Thus, instead of saying that the area of a rectangle with base 4 in. and height 2 in. is 4×2 sq. in., we express a general law by saying that $a = bh$. Algebra, therefore, is a generalized arithmetic, and the equation is the chief object of attention.

Algebra also enters into all subsequent branches of mathematics, and its relation to trigonometry will be found to be very close.

3. The Nature of Geometry. In geometry we study the forms and relations of figures, proving many properties and effecting numerous constructions concerning them.

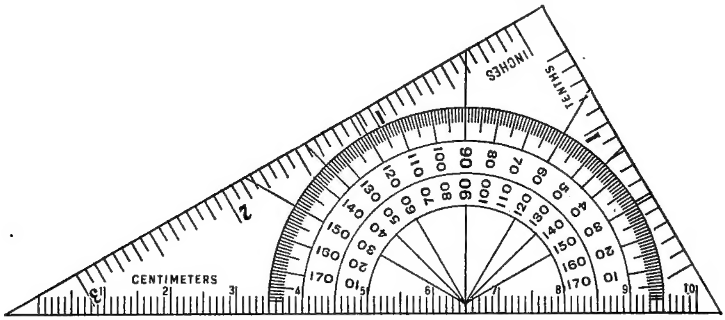
Geometry, like algebra and arithmetic, enters into the work in trigonometry. Indeed, trigonometry may almost be said to unite arithmetic, algebra, and geometry in one subject.

4. The Nature of Trigonometry. We are now about to begin another branch of mathematics, one not chiefly relating to numbers although it uses numbers, not primarily devoted to equations although using equations, and not concerned principally with the study of geometric forms although freely drawing upon the facts of geometry.

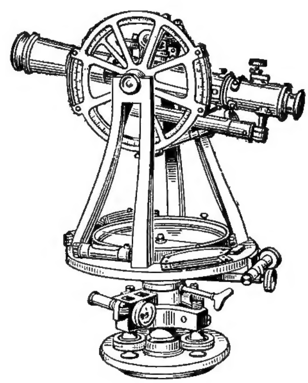
Trigonometry is concerned chiefly with the relation of certain lines in a triangle (*trigon*, "a triangle," + *metrein*, "to measure") and forms the basis of the mensuration used in surveying, engineering, mechanics, geodesy, and astronomy.

5. How Angles are Measured. For ordinary purposes angles can be measured with a protractor to a degree of accuracy of about 30′.

The student will find it advantageous to use the convenient protractor furnished with this book and shown in the illustration below.



For work out of doors it is customary to use a transit, an instrument by means of which angles can be measured to minutes. By turning the top of the transit to the right or left, horizontal angles can be measured on the horizontal plate. By turning the telescope up or down, vertical angles can be measured on the vertical circle seen in the illustration.



For astronomical purposes, where great care is necessary in measuring angles, large circles are used.

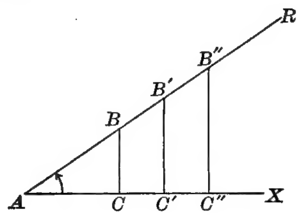
The degree of accuracy required in measuring an angle depends upon the nature of the problem. We shall now assume that we can measure angles in degrees, minutes, and seconds, or in degrees and decimal parts of a degree. Thus 15° 30′ is the same as 15.5°, and 15° 30′ 36″ is the same as $15\frac{1}{2}° + \frac{36}{3600}$ of 1°, or 15.51°.

The ancient Greek astronomers had no good symbols for fractions. The best system they could devise for close approximations was the so-called sexagesimal one, in which there appear only the numerators of fractions whose denominators are powers of 60. This system seems to have been first suggested by the Babylonians, but to have been developed by the Greeks. It is much inferior to the decimal system that was perfected about 1600, but the world still continues to use it for the measure of angles and time. The decimal division of the angle is, however, gaining ground, and in due time will probably replace the more cumbersome one with which we are familiar.

In this book we shall use both the ancient and modern systems, but with the chief attention to the former, since this is still the more common.

6. Functions of an Angle. In the annexed figure, if the line AR moves about the point A in the sense indicated by the arrow, from the position AX as an initial position, it generates the angle A.

If from the points $B, B', B'', \ldots$, on AR, we let fall the perpendiculars BC, $B'C'$, $B''C''$, ..., on AX, we form a series of similar triangles ACB, $AC'B'$, $AC''B''$, and so on. The corresponding sides of these triangles are proportional. That is,



$$\frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''} = \cdots;$$

$$\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{B''C''}{AC''} = \cdots;$$

$$\frac{AB}{AC} = \frac{AB'}{AC'} = \frac{AB''}{AC''} = \cdots;$$

and similarly for the ratios

$$\frac{AB}{BC}, \quad \frac{AC}{BC}, \quad \frac{AC}{AB},$$

each of which has a series of other ratios equal to it.

For example, $$\frac{AB}{BC} = \frac{AB'}{B'C'} = \frac{AB''}{B''C''}.$$

That is, *these ratios remain unchanged so long as the angle remains unchanged, but they change as the angle changes.*

Each of the above ratios is therefore a *function* of the angle A.

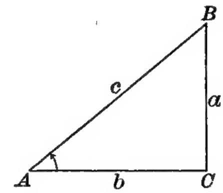
As already learned in algebra and geometry, a magnitude which depends upon another magnitude for its value is called a *function* of the latter magnitude. Thus a circle is a function of the radius, the area of a square is a function of the side, the surface of a sphere is a function of the diameter, and the volume of a pyramid is a function of the base and altitude.

We indicate a function of x by such symbols as $f(x)$, $F(x)$, $f'(x)$, and $\phi(x)$, and we read these "f of x, f-major of x, f-prime of x, and phi of x" respectively.

For example, if we are repeatedly using some long expression like $x^4 + 3x^3 - 2x^2 + 7x - 4$, we may speak of it briefly as $f(x)$. If we are using some function of angle A, we may designate this as $f(A)$. If we wish to speak of some other function of A, we may write it $f'(A)$, $F(A)$, or $\phi(A)$.

In trigonometry we shall make much use of various functions of an angle, but we shall give to them special names and symbols. On this account the ordinary function symbols of algebra, mentioned above, will not be used frequently in trigonometry, but they will be used often enough to make it necessary that the student should understand their significance.

7. The Six Functions. Since with a given angle A we may take any one of the triangles described in § 6, we shall consider the triangle ACB, lettered as here shown.



It has long been the custom to letter in this way the hypotenuse, sides, and angles of the first triangle considered in trigonometry, C being the right angle, and the hypotenuse and sides bearing the small letters corresponding to the opposite capitals. By the *sides* of the triangle is meant the sides a and b, c being called the hypotenuse. The sides a and b are also called the *legs* of the triangle, particularly by early writers, since it was formerly the custom to represent the triangle as standing on the hypotenuse.

The ratios $\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}$, and $\frac{c}{a}$ have the following names:

$\frac{a}{c}$ is called the *sine* of A, written $\sin A$;

$\frac{b}{c}$ is called the *cosine* of A, written $\cos A$;

$\frac{a}{b}$ is called the *tangent* of A, written $\tan A$;

$\frac{b}{a}$ is called the *cotangent* of A, written $\cot A$;

$\frac{c}{b}$ is called the *secant* of A, written $\sec A$;

$\frac{c}{a}$ is called the *cosecant* of A, written $\csc A$.

That is,

$$\sin A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}, \quad \cos A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}},$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}, \quad \cot A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}},$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}, \quad \csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}.$$

These definitions must be thoroughly learned, since they are the foundation upon which the whole science is built. The student should practice upon them, with the figure before him, until he can tell instantly what ratio is meant by $\sec A$, $\cot A$, $\sin A$, and so on, in whatever order these functions are given.

There are also two other functions, rarely used at present. These are the versed sine $A = 1 - \cos A$, and the coversed sine $A = 1 - \sin A$. These definitions need not be learned at this time, since they will be given again when the functions are met later in the work.

Exercise 1. The Six Functions

1. In the figure of § 7, $\sin B = \frac{b}{c}\cdot$ Write the other five functions of the angle B.

2. Show that in the right triangle ACB (§ 7) the following relations exist:

$$\sin A = \cos B,\quad \cos A = \sin B,\quad \tan A = \cot B,$$
$$\cot A = \tan B,\quad \sec A = \csc B,\quad \csc A = \sec B.$$

State which of the following is the greater:

3. $\sin A$ or $\tan A$.
4. $\cos A$ or $\cot A$.
5. $\sec A$ or $\tan A$.
6. $\csc A$ or $\cot A$.

Find the values of the six functions of A, if a, b, c respectively have the following values:

7. 3, 4, 5.
8. 5, 12, 13.
9. 8, 15, 17.
10. 9, 40, 41.
11. 3.9, 8, 8.9.
12. 1.19, 1.20, 1.69.

13. What condition must be fulfilled by the lengths of the three lines a, b, c (§ 7) to make them the sides of a right triangle? Show that this condition is fulfilled in Exs. 7–12.

Find the values of the six functions of A, if a, b, c respectively have the following values:

14. $2n,\ n^2-1,\ n^2+1.$
15. $n,\ \frac{n^2-1}{2},\ \frac{n^2+1}{2}.$
16. $2mn,\ m^2-n^2,\ m^2+n^2.$
17. $\frac{2mn}{m-n},\ m+n,\ \frac{m^2+n^2}{m-n}.$

18. As in Ex. 13, show that the condition for a right triangle is fulfilled in Exs. 14–17.

Given $a^2+b^2=c^2$, find the six functions of A when:

19. $a=b$.
20. $a=2b$.
21. $a=\frac{2}{3}c$.

Given $a^2+b^2=c^2$, find the six functions of B when:

22. $a=24,\ b=143$.
23. $b=9.5,\ c=19.3$.
24. $a=0.264,\ c=0.265$.
25. $b=2\sqrt{pq},\ c=p+q$.

Given $a^2+b^2=c^2$, find the six functions of A and also the six functions of B when:

26. $a=\sqrt{p^2+q^2},\ b=\sqrt{2pq}$.
27. $a=\sqrt{p^2+p},\ c=p+1$.

In the right triangle ACB, as shown in § 7:

28. Find the length of side a if $\sin A = \frac{3}{5}$, and $c = 20.5$.

29. Find the length of side b if $\cos A = 0.44$, and $c = 3.5$.

30. Find the length of side a if $\tan A = 3\frac{2}{3}$, and $b = 2\frac{5}{11}$.

31. Find the length of side b if $\cot A = 4$, and $a = 1700$.

32. Find the length of the hypotenuse if $\sec A = 2$, and $b = 2000$.

33. Find the length of the hypotenuse if $\csc A = 6.4$, and $a = 35.6$.

Find the hypotenuse and other side of a right triangle, given:

34. $b = 6$, $\tan A = \frac{3}{2}$.

35. $a = 3.5$, $\cos A = 0.5$.

36. $b = 4$, $\csc A = 1\frac{2}{3}$.

37. $b = 2$, $\sin A = 0.6$.

38. The hypotenuse of a right triangle is 2.5 mi., $\sin A = 0.6$, and $\cos A = 0.8$. Compute the sides of the triangle.

39. Construct with a protractor the angles 20°, 40°, and 70°; determine their functions by measuring the necessary lines and compare the values obtained in this way with the more nearly correct values given in the following table:

	sin	cos	tan	cot	sec	csc
20°	0.342	0.940	0.364	2.747	1.064	2.924
40°	0.643	0.766	0.839	1.192	1.305	1.556
70°	0.940	0.342	2.747	0.364	2.924	1.064

Find, by means of the above table, the sides and hypotenuse of a right triangle, given:

40. $A = 20°$, $c = 1$.

41. $A = 20°$, $c = 4$.

42. $A = 20°$, $c = 3.5$.

43. $A = 20°$, $c = 4.8$.

44. $A = 20°$, $c = 7\frac{1}{2}$.

45. $A = 40°$, $c = 1$.

46. $A = 40°$, $c = 3$.

47. $A = 40°$, $c = 7$.

48. $A = 40°$, $c = 10.7$.

49. $A = 40°$, $c = 250$.

50. $A = 70°$, $c = 2$.

51. $A = 70°$, $a = 2$.

52. $A = 70°$, $b = 2$.

53. $A = 70°$, $a = 25$.

54. $A = 70°$, $b = 150$.

55. By dividing the length of a vertical rod by the length of its horizontal shadow, the tangent of the angle of elevation of the sun at that time was found to be 0.82. How high is a tower, if the length of its horizontal shadow at the same time is 174.3 yd.?

56. A pin is stuck upright on a table top and extends upward 1 in. above the surface. When its shadow is $\frac{7}{8}$ in. long, what is the tangent of the angle of elevation of the sun? How high is a telegraph pole whose horizontal shadow at that instant is 21 ft.?

8. Functions of Complementary Angles. In the annexed figure we see that B is the complement of A; that is, $B = 90° - A$. Hence,

$$\sin A = \frac{a}{c} = \cos B = \cos(90° - A),$$

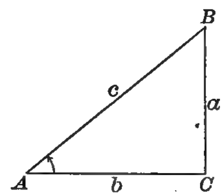
$$\cos A = \frac{b}{c} = \sin B = \sin(90° - A),$$

$$\tan A = \frac{a}{b} = \cot B = \cot(90° - A),$$

$$\cot A = \frac{b}{a} = \tan B = \tan(90° - A),$$

$$\sec A = \frac{c}{b} = \csc B = \csc(90° - A),$$

$$\csc A = \frac{c}{a} = \sec B = \sec(90° - A).$$



That is, *each function of an acute angle is equal to the co-named function of the complementary angle.*

Co-sine means *complement's sine*, and similarly for the other *co*-functions.

It is therefore seen that $\sin 75° = \cos(90° - 75°) = \cos 15°$, $\sec 82° 30' = \csc(90° - 82° 30') = \csc 7° 30'$, and so on.

Therefore, *any function of an angle between 45° and 90° may be found by taking the co-named function of the complementary angle, which is between 0° and 45°.*

Hence, we need never have a direct table of functions beyond 45°. We shall presently see (§ 12) that this is of great advantage.

Exercise 2. Functions of Complementary Angles

Express as functions of the complementary angle:

1. $\sin 30°$.	5. $\sin 50°$.	9. $\sin 60°$.	13. $\sin 75° 30'$.
2. $\cos 20°$.	6. $\tan 60°$.	10. $\cos 60°$.	14. $\tan 82° 45'$.
3. $\tan 40°$.	7. $\sec 75°$.	11. $\tan 45°$.	15. $\sec 68° 15'$.
4. $\sec 25°$.	8. $\csc 85°$.	12. $\sec 45°$.	16. $\cos 88° 10'$.

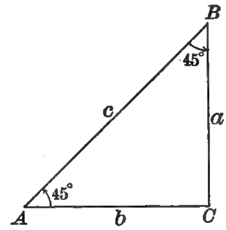
Express as functions of an angle less than 45°:

17. $\sin 65°$.	20. $\cos 52°$.	23. $\sin 89°$.	26. $\sin 77\frac{1}{2}°$.
18. $\tan 80°$.	21. $\cot 61°$.	24. $\cos 86°$.	27. $\cos 82\frac{1}{2}°$.
19. $\sec 77°$.	22. $\csc 78°$.	25. $\sec 88°$.	28. $\tan 88.6°$.

Find A, given the following relations:

29. $90° - A = A$.	31. $90° - A = 2A$.
30. $\cos A = \sin A$.	32. $\cos A = \sin 2A$.

9. Functions of 45°. The functions of certain angles, among them 45°, are easily found. In the isosceles right triangle ACB we have $A = B = 45°$, and $a = b$. Furthermore, since $a^2 + b^2 = c^2$, we have $2a^2 = c^2$, $a\sqrt{2} = c$, and $a = \frac{1}{2}c\sqrt{2}$. Hence,



$$\sin 45° = \cos 45° = \frac{\frac{1}{2}c\sqrt{2}}{c} = \frac{1}{2}\sqrt{2};$$

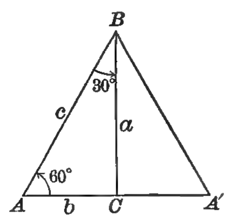
$$\tan 45° = \cot 45° = \frac{a}{b} = 1;$$

$$\sec 45° = \csc 45° = \frac{a\sqrt{2}}{a} = \sqrt{2}.$$

We have therefore found all six functions of 45°. For purposes of computation these are commonly expressed as decimal fractions. Since $\sqrt{2} = 1.4142+$, we have the following values:

$\sin 45° = 0.7071$,	$\cos 45° = 0.7071$,
$\tan 45° = 1$,	$\cot 45° = 1$,
$\sec 45° = 1.4142$,	$\csc 45° = 1.4142$.

10. Functions of 30° and 60°. In the equilateral triangle $AA'B$ here shown, BC is the perpendicular bisector of the base. Also, $b = \frac{1}{2}c$, and $a = \sqrt{c^2 - b^2} = \sqrt{c^2 - \frac{1}{4}c^2} = \frac{1}{2}c\sqrt{3}$. Hence,



$$\sin 30° = \cos 60° = \frac{b}{c} = \frac{1}{2};$$

$$\cos 30° = \sin 60° = \frac{a}{c} = \frac{1}{2}\sqrt{3};$$

$$\tan 30° = \cot 60° = \frac{b}{a} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3};$$

$$\cot 30° = \tan 60° = \frac{a}{b} = \sqrt{3};$$

$$\sec 30° = \csc 60° = \frac{c}{a} = \frac{c}{\frac{1}{2}c\sqrt{3}} = \frac{2}{3}\sqrt{3};$$

$$\csc 30° = \sec 60° = \frac{c}{b} = 2.$$

The sine and cosine of 30°, 45°, and 60° are easily remembered, thus:

$$\sin 30° = \tfrac{1}{2}\sqrt{1}, \quad \sin 45° = \tfrac{1}{2}\sqrt{2}, \quad \sin 60° = \tfrac{1}{2}\sqrt{3};$$
$$\cos 30° = \tfrac{1}{2}\sqrt{3}, \quad \cos 45° = \tfrac{1}{2}\sqrt{2}, \quad \cos 60° = \tfrac{1}{2}\sqrt{1}.$$

The functions of other angles are not so easily computed. The computation requires a study of series and is explained in more advanced works on mathematics. For the present we assume that the functions of all angles have been computed and are available, as is really the case.

Exercise 3. Functions of 30°, 45°, and 60°

Given $\sqrt{3} = 1.7320$, express as decimal fractions the following:

1. $\sin 30°$.
2. $\cos 30°$.
3. $\tan 30°$.
4. $\cot 30°$.
5. $\sec 30°$.
6. $\csc 30°$.
7. $\sin 60°$.
8. $\cos 60°$.
9. $\tan 60°$.
10. $\cot 60°$.
11. $\sec 60°$.
12. $\csc 60°$.

Write the ratios of the following, simplifying the results:

13. $\sin 45°$ to $\sin 30°$.
14. $\cos 45°$ to $\cos 30°$.
15. $\tan 45°$ to $\tan 30°$.
16. $\cot 45°$ to $\cot 30°$.
17. $\sec 45°$ to $\sec 30°$.
18. $\csc 45°$ to $\csc 30°$.
19. $\sin 30°$ to $\sin 60°$.
20. $\cos 30°$ to $\cos 60°$.
21. $\tan 30°$ to $\tan 60°$.
22. $\cot 30°$ to $\cot 60°$.
23. $\sec 30°$ to $\sec 60°$.
24. $\csc 30°$ to $\csc 60°$.

Express as functions of angles less than 45°:

25. $\sin 62° 17' 40''$.
26. $\tan 75° 28' 35''$.
27. $\sec 87° 32' 51''$.
28. $\cos 88° 0' 27''$.
29. $\sin 75.8°$.
30. $\cos 82.75°$.
31. $\tan 68.82°$.
32. $\sec 85.95°$.

Find A, given the following relations:

33. $90° - A = 45° - \frac{1}{2}A$.
34. $90° - \frac{1}{2}A = A$.
35. $45° + A = 90° - A$.
36. $90° - 4A = A$.
37. $90° - A = nA$.
38. $\cos A = \sin(45° - \frac{1}{2}A)$.
39. $\cot \frac{1}{2}A = \tan A$.
40. $\tan(45° + A) = \cot A$.
41. $\cos 4A = \sin A$.
42. $\cot A = \tan nA$.
43. By what must $\sin 45°$ be multiplied to equal $\tan 30°$?
44. By what must $\sec 45°$ be multiplied to equal $\csc 30°$?
45. By what must $\cos 45°$ be multiplied to equal $\tan 60°$?
46. By what must $\csc 60°$ be divided to equal $\tan 45°$?
47. By what must $\csc 30°$ be divided to equal $\tan 30°$?
48. What is the ratio of $\sin 45° \sec 45°$ to $\cos 60°$?
49. What is the ratio of $\cos 45° \csc 45°$ to $\cos 30° \csc 30°$?
50. What is the ratio of $\sin 45° \sin 30°$ to $\cos 45° \cos 30°$?
51. What is the ratio of $\tan 30° \cot 30°$ to $\tan 60° \cot 60°$?
52. From the statement $\tan 30° = \frac{1}{3}\sqrt{3}$ find $\cot 60°$.

11. Values of the Functions. The values of the functions have been computed and tables constructed giving these values. One of these tables is shown on page 11 and will suffice for the work required on the next few pages.

This table gives the values of the functions to four decimal places for every degree from 0° to 90°. All such values are only approximate, the values of the functions being, in general, incommensurable with unity and not being expressible by means of common fractions or by means of decimal fractions with a finite number of decimal places.

12. Arrangement of the Table. As explained in § 8, $\cos 45° = \sin 45°$, $\cos 46° = \sin 44°$, $\cos 47° = \sin 43°$, and so on. Hence the column of sines from 0° to 45° is the same as the column of cosines from 45° to 90°. Therefore

In finding the functions of angles from 0° to 45° read from the top down; in finding the functions of angles from 45° to 90° read from the bottom up.

Exercise 4. Use of the Table

From the table on page 11 find the values of the following:

1. $\sin 5°$.	9. $\cos 6°$.	17. $\cot 5°$.	25. $\sec 0°$.
2. $\sin 14°$.	10. $\sin 84°$.	18. $\tan 85°$.	26. $\csc 90°$.
3. $\sin 21°$.	11. $\cos 14°$.	19. $\cot 11°$.	27. $\sec 15°$.
4. $\sin 30°$.	12. $\sin 76°$.	20. $\tan 79°$.	28. $\csc 75°$.
5. $\cos 85°$.	13. $\cos 24°$.	21. $\tan 21°$.	29. $\csc 12°$.
6. $\cos 76°$.	14. $\sin 66°$.	22. $\cot 69°$.	30. $\sec 78°$.
7. $\cos 69°$.	15. $\cos 35°$.	23. $\tan 45°$.	31. $\csc 44°$.
8. $\cos 60°$.	16. $\sin 55°$.	24. $\cot 45°$.	32. $\sec 46°$.

33. Find the difference between $2 \sin 9°$ and $\sin(2 \times 9°)$.

34. Find the difference between $3 \tan 5°$ and $\tan(3 \times 5°)$.

35. Which is the larger, $2 \sec 10°$ or $\sec(2 \times 10°)$?

36. Which is the larger, $2 \csc 10°$ or $\csc(2 \times 10°)$?

37. Which is the larger, $2 \cos 15°$ or $\cos(2 \times 15°)$?

38. Compare $3 \sin 20°$ with $\sin(3 \times 20°)$; with $\sin(2 \times 20°)$.

39. Compare $3 \tan 10°$ with $\tan(3 \times 10°)$; with $\tan(2 \times 10°)$.

40. Compare $3 \cos 10°$ with $\cos(3 \times 10°)$; with $\cos(2 \times 10°)$.

41. Is $\sin(10° + 20°)$ equal to $\sin 10° + \sin 20°$?

42. When the angle is increased from 0° to 90° which of the six functions are increased and which are decreased?

TABLE OF TRIGONOMETRIC FUNCTIONS FOR EVERY DEGREE FROM 0° TO 90°

Angle	sin	cos	tan	cot	sec	csc	
0°	.0000	1.0000	.0000	∞	1.0000	∞	**90°**
1°	.0175	.9998	.0175	57.2900	1.0002	57.2987	89°
2°	.0349	.9994	.0349	28.6363	1.0006	28.6537	88°
3°	.0523	.9986	.0524	19.0811	1.0014	19.1073	87°
4°	.0698	.9976	.0699	14.3007	1.0024	14.3356	86°
5°	.0872	.9962	.0875	11.4301	1.0038	11.4737	**85°**
6°	.1045	.9945	.1051	9.5144	1.0055	9.5668	84°
7°	.1219	.9925	.1228	8.1443	1.0075	8.2055	83°
8°	.1392	.9903	.1405	7.1154	1.0098	7.1853	82°
9°	.1564	.9877	.1584	6.3138	1.0125	6.3925	81°
10°	.1736	.9848	.1763	5.6713	1.0154	5.7588	**80°**
11°	.1908	.9816	.1944	5.1446	1.0187	5.2408	79°
12°	.2079	.9781	.2126	4.7046	1.0223	4.8097	78°
13°	.2250	.9744	.2309	4.3315	1.0263	4.4454	77°
14°	.2419	.9703	.2493	4.0108	1.0306	4.1336	76°
15°	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75°**
16°	.2756	.9613	.2867	3.4874	1.0403	3.6280	74°
17°	.2924	.9563	.3057	3.2709	1.0457	3.4203	73°
18°	.3090	.9511	.3249	3.0777	1.0515	3.2361	72°
19°	.3256	.9455	.3443	2.9042	1.0576	3.0716	71°
20°	.3420	.9397	.3640	2.7475	1.0642	2.9238	**70°**
21°	.3584	.9336	.3839	2.6051	1.0711	2.7904	69°
22°	.3746	.9272	.4040	2.4751	1.0785	2.6695	68°
23°	.3907	.9205	.4245	2.3559	1.0864	2.5593	67°
24°	.4067	.9135	.4452	2.2460	1.0946	2.4586	66°
25°	.4226	.9063	.4663	2.1445	1.1034	2.3662	**65°**
26°	.4384	.8988	.4877	2.0503	1.1126	2.2812	64°
27°	.4540	.8910	.5095	1.9626	1.1223	2.2027	63°
28°	.4695	.8829	.5317	1.8807	1.1326	2.1301	62°
29°	.4848	.8746	.5543	1.8040	1.1434	2.0627	61°
30°	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60°**
31°	.5150	.8572	.6009	1.6643	1.1666	1.9416	59°
32°	.5299	.8480	.6249	1.6003	1.1792	1.8871	58°
33°	.5446	.8387	.6494	1.5399	1.1924	1.8361	57°
34°	.5592	.8290	.6745	1.4826	1.2062	1.7883	56°
35°	.5736	.8192	.7002	1.4281	1.2208	1.7434	**55°**
36°	.5878	.8090	.7265	1.3764	1.2361	1.7013	54°
37°	.6018	.7986	.7536	1.3270	1.2521	1.6616	53°
38°	.6157	.7880	.7813	1.2799	1.2690	1.6243	52°
39°	.6293	.7771	.8098	1.2349	1.2868	1.5890	51°
40°	.6428	.7660	.8391	1.1918	1.3054	1.5557	**50°**
41°	.6561	.7547	.8693	1.1504	1.3250	1.5243	49°
42°	.6691	.7431	.9004	1.1106	1.3456	1.4945	48°
43°	.6820	.7314	.9325	1.0724	1.3673	1.4663	47°
44°	.6947	.7193	.9657	1.0355	1.3902	1.4396	46°
45°	.7071	.7071	1.0000	1.0000	1.4142	1.4142	**45°**
	cos	**sin**	**cot**	**tan**	**csc**	**sec**	**Angle**

13. Reciprocal Functions. Considering the definitions of the six functions, we see that, since

$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b},$$

$$\csc A = \frac{c}{a}, \qquad \sec A = \frac{c}{b}, \qquad \cot A = \frac{b}{a},$$

The sine is the reciprocal of the cosecant, the cosine is the reciprocal of the secant, and the tangent is the reciprocal of the cotangent.

That is,

$$\sin A = \frac{1}{\csc A}, \qquad \cos A = \frac{1}{\sec A}, \qquad \tan A = \frac{1}{\cot A},$$

$$\csc A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}, \qquad \cot A = \frac{1}{\tan A}.$$

Hence $\sin A \csc A = 1$, $\cos A \sec A = 1$, and $\tan A \cot A = 1$. For example, from the table on page 11 we find $\sin 27° \csc 27°$ thus:

$$\sin 27° = 0.4540.$$

$$\csc 27° = 2.2027.$$

Therefore $\qquad \sin 27° \csc 27° = 0.4540 \times 2.2027$

$$= 1.00002580, \text{ or approximately } 1.$$

We have shown that $\sin A \csc A = 1$ exactly, but the numbers given in the table are, as before stated, correct only to four decimal places.

Exercise 5. Use of the Table

Using the values given in the table on page 11, show as above that the following are reciprocals:

1. $\sin 30°$, $\csc 30°$.
2. $\sin 25°$, $\csc 25°$.
3. $\cos 35°$, $\sec 35°$.
4. $\sin 10°$, $\csc 10°$.
5. $\tan 10°$, $\cot 10°$.
6. $\cos 10°$, $\sec 10°$.
7. $\sin 75°$, $\csc 75°$.
8. $\cos 75°$, $\sec 75°$.
9. $\tan 75°$, $\cot 75°$.

10. From the table show that the ratio of $\sin 20° \csc 20°$ to $\tan 50° \cot 50°$ is 1.

11. Similarly, show that $\cos 40° \sec 40° : \tan 70° \cot 70° = 1$.

In the right triangle ACB, as shown in § 7:

12. Find the length of side a if $A = 30°$, and $c = 75.2$.
13. Find the length of side a if $A = 45°$, and $c = 1.414$.
14. Find the length of side b if $A = 30°$, and $c = 115.47$.
15. Find the length of side a if $A = 60°$, and $b = 34.64$.
16. Find the length of side b if $A = 60°$, and $c = 25.72$.
17. Find the length of side a if $A = 30°$, and $c = 45.28$.

14. Other Relations of Functions. Since, from the figure in § 7, $a^2 + b^2 = c^2$, we have

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1,$$

or

$$\sin^2 A + \cos^2 A = 1.$$

It is customary to write $\sin^2 A$ for $(\sin A)^2$, and similarly for the other functions.

This formula is one of the most important in trigonometry and should be memorized. From it we see that

$$\sin A = \sqrt{1 - \cos^2 A}, \qquad \cos A = \sqrt{1 - \sin^2 A}.$$

Furthermore, since $\tan A = \frac{a}{b}$, $\sin A = \frac{a}{c}$, and $\cos A = \frac{b}{c}$, it follows that

$$\tan A = \frac{\sin A}{\cos A}.$$

This is also an important formula to be memorized. From it we see that $\tan A \cos A = \sin A$, and, in general, that we can find any one of the functions, sine, cosine, or tangent, given the other two.

Furthermore, from the same equation $a^2 + b^2 = c^2$ we see that $1 + \frac{a^2}{b^2} = \frac{c^2}{b^2}$. Hence we see that

$$1 + \tan^2 A = \sec^2 A.$$

In a similar manner we may prove that $1 + \frac{b^2}{a^2} = \frac{c^2}{a^2}$; whence we have the formula

$$1 + \cot^2 A = \csc^2 A.$$

These two formulas should be memorized.

From these formulas the following relations can easily be deduced:

$$\sin x = \cos x \tan x = \cos x/\cot x = \tan x/\sec x.$$
$$\cos x = \cot x \sin x = \cot x/\csc x = \sin x/\tan x.$$
$$\tan x = \sin x \sec x = \sin x/\cos x = \sec x/\csc x.$$
$$\cot x = \csc x \cos x = \csc x/\sec x = \cos x/\sin x.$$
$$\sec x = \tan x \csc x = \tan x/\sin x = \csc x/\cot x.$$
$$\csc x = \sec x \cot x = \sec x/\tan x = \cot x/\cos x.$$

It is often convenient to recall these relations, and this can be done by the aid of a simple mnemonic:

$$\begin{array}{ccc} & \tan x & \\ \sin x & & \sec x \\ \cos x & & \csc x \\ & \cot x & \end{array}$$

In the above diagram, *any function is equal to the product of the two adjacent functions, or to the quotient of either adjacent function divided by the one beyond it.*

15. Practical Use of the Sine. Since by definition we have

$$\frac{a}{c} = \sin A,$$

we see that $$a = c \sin A.$$

We might also derive the equation

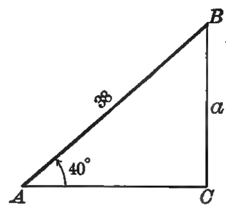
$$c = \frac{a}{\sin A}.$$

But since $\frac{1}{\sin A} = \csc A$ (§ 13), it is easier at present to use

$$c = a \csc A,$$

and this will be considered when we come to study the cosecant.

1. Given $c = 38$ and $A = 40°$, find a.



As above, $$a = c \sin A.$$

From the table,

$$\begin{aligned} \sin 40° &= 0.6428 \\ \text{and}\quad c &= \underline{38} \\ & 5\,1424 \\ & \underline{19\,284} \\ c \sin A &= 24.4264 \end{aligned}$$

But since the table on page 11 gives only the first four figures of sin 40°, *we can expect only the first four figures of the result to be correct.* We therefore say that $a = 24.43 -$. If the third decimal place were less than 5, the value of a would be written $24.42 +$.

Some check should always be applied to the result. In this case we may proceed as follows: $24.4264 \div 38 = 0.6428$, which is sin 40°.

2. Given $c = 10$ and $a = 6.293$, find A.

Since $$\frac{a}{c} = \sin A,$$

we have $$\frac{6.293}{10} = 0.6293 = \sin A.$$

Looking in the table we see that

$$0.6293 = \sin 39°;$$

whence $$A = 39°.$$

3. Given $a = 4.68\frac{1}{4}$ and $A = 22°$, find c.

As stated above, c may be found from the formula $a = c \sin A$ by using a and $\sin A$, although we shall later use the cosecant for this purpose. Substituting the given values, we have

$$4.68\tfrac{1}{4} = c \sin 22°,$$

or $$4.6825 = 0.3746\, c.$$

Dividing by 0.3746, $$12.5 = c.$$

What check should be applied here and in Ex. 2?

Exercise 6. Use of the Sine

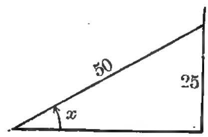
Find a to four figures, given the following:

1. $c=10$, $A=10°$.
2. $c=15$, $A=15°$.
3. $c=58$, $A=45°$.
4. $c=75$, $A=50°$.

Find A, given the following:

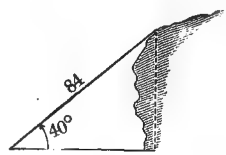
5. $c=10$, $a=2.079$.
6. $c=20$, $a=6.840$.
7. $c=2$, $a=1.2586$.
8. $c=50$, $a=34.1$.

9. A 50-foot ladder resting against the side of a house reaches a point 25 ft. from the ground. What angle does it make with the ground?

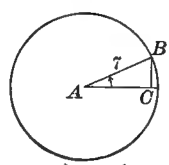


In all such cases the ground should be considered level and the side of the building should be considered vertical unless the contrary is expressly stated.

10. From the top of a rock a cord is stretched to a point on the ground, making an angle of 40° with the horizontal plane. The cord is 84 ft. long. Assuming the cord to be straight, how high is the rock?

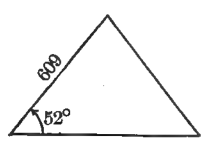


11. Find the side of a regular decagon inscribed in a circle of radius 7 ft.

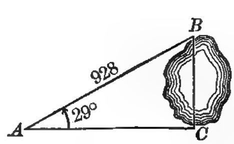


What is the central angle? What is half of this angle? Find BC and double it. By this plan we can find the perimeter of any inscribed regular polygon, given the radius of the circle. In this way we could approximate the value of π. For example, we see that the semiperimeter of a polygon of 90 sides in a unit circle is $90 \times \sin 2°$, or 90×0.0349, or 3.141.

12. The edge of the Great Pyramid is 609 ft. and makes an angle of 52° with the horizontal plane. What is the height of the pyramid?



13. Wishing to measure BC, the length of a pond, a surveyor ran a line CA at right angles to BC. He measured AB and $\angle A$, finding that $AB=928$ ft., and $A=29°$. Find the length of BC.

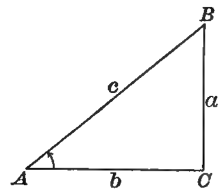


In practical surveying we would probably use an oblique triangle, although the work as given here is correct. The oblique triangle is considered later.

16. Practical Use of the Cosine. Since by definition we have

$$\frac{b}{c} = \cos A,$$

we see that $$b = c \cos A.$$



1. Given $c = 28$ and $A = 46°$, find b.

From the table, $\cos 46° = 0.6947$

and

$$\begin{array}{r} c = \quad 28 \\ \hline 5\,5576 \\ 13\,894 \\ \hline 19.4516 \end{array}$$

Hence, to four figures, $b = 19.45$.

2. Given $c = 2$ and $b = 1.9022$, find A.

Since $$\frac{b}{c} = \cos A,$$

we have $$1.9022 \div 2 = 0.9511 = \cos A.$$

From the table, $0.9511 = \cos 18°$.

Hence $A = 18°$.

What check should be applied here and in Ex. 1?

Exercise 7. Use of the Cosine

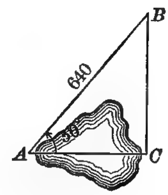
Find b to four figures, given the following:

1. $c = 11$, $A = 10°$.
2. $c = 14$, $A = 16°$.
3. $c = 28$, $A = 24°$.
4. $c = 41$, $A = 39°$.
5. $c = 75$, $A = 42°$.
6. $c = 2.8$, $A = 48°$.
7. $c = 9.7$, $A = 52°$.
8. $c = 11.2$, $A = 58°$.
9. $c = 12.5$, $A = 67°$.
10. $c = 28.25$, $A = 75°$.

Find A, given the following:

11. $c = 10$, $b = 9.848$.
12. $c = 20$, $b = 19.126$.
13. $c = 40$, $b = 35.952$.
14. $c = 17.6$, $b = 8.8$.
15. $c = 500$, $b = 227$.
16. $c = 600$, $b = 205.2$.
17. $c = 200$, $b = 117.56$.
18. $c = 187$, $b = 93\frac{1}{2}$.
19. $c = 300$, $b = 102\frac{3}{5}$.
20. $c = 1000$, $b = 104\frac{1}{2}$.

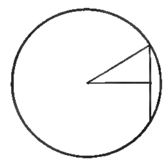
21. A flagstaff breaks off 22 ft. from the top and, the parts still holding together, the top of the staff reaches the earth 11 ft. from the foot. What angle does it make with the ground?



22. Wishing to measure the length of a pond, a class constructed a right triangle as shown in the figure. If $AB = 640$ ft. and $A = 50°$, required the distance AC.

23. In the same figure what is the length of AC when $AB = 500$ ft. and $A = 40°$?

24. In the same figure, if $AC = 731.4$ ft. and $AB = 1000$ ft., what is the size of angle A?



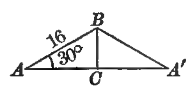
25. A regular hexagon is inscribed in a circle of radius 9 in. How far is it from the center to a side?

Having found this distance, the *apothem*, and knowing that a side of the regular hexagon equals the radius, we can find the area, as required in Ex. 26.

26. What is the area of a regular hexagon inscribed in a circle of radius 8 in.?

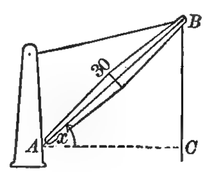
27. A ship sails northeast 8 mi. It is then how many miles to the east of the starting point?

Northeast is 45° east of north. In all such cases in plane trigonometry the figure is supposed to be a plane. For long distances it would be necessary to consider a spherical triangle.



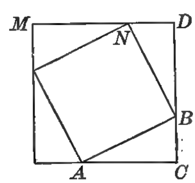
28. Some 16-foot roof timbers make an angle of 30° with the horizontal in an A-shaped roof, as shown in the figure. Find AA', the span of the roof.

29. An equilateral triangle is inscribed in a circle of radius 12 in. How far is it from the center to a side?



30. A crane AB, 30 ft. long, makes an angle of x degrees with the horizontal line AC. Find the distance AC when $x = 20$; when $x = 45$; when $x = 65$; when $x = 0$; when $x = 90$.

31. In Ex. 30 what angle does the crane make with the horizontal when $AC = 15$ ft.? when $AC = 30$ ft.?



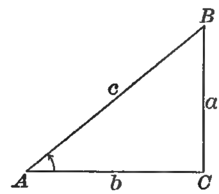
32. The square AN, of which the side is 200 ft., is inscribed in the square CM. AC is 181.26 ft. Required the angles that the sides of the small square make with the large one.

33. In Ex. 32 find the required angles when $AB = 15$ in. and $BC = 7\frac{1}{2}$ in.; when $AB = 20$ in. and $BC = 10.3$ in.

34. The edge of the Great Pyramid is 609 ft., and it makes an angle of 52° with the horizontal plane. What is the diagonal of the base?

17. Practical Use of the Tangent. Since by definition we have

$$\frac{a}{b} = \tan A,$$



we see that $a = b \tan A.$

Given $b = 12$ and $A = 35°$, find a.

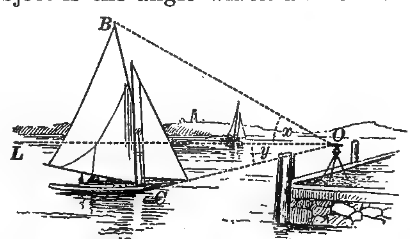
From the table,

$$\begin{array}{rr} \tan 35° = & 0.7002 \\ b = & 12 \\ \hline & 1\ 4004 \\ & 7\ 002 \\ \hline & 8.4024 \end{array}$$

Hence, to four figures, $a = 8.402.$

The figures 1, 2, $\cdots$, 9 are often spoken of as *significant figures*. In 8.402 the zero is, however, looked upon as a significant figure, but not in a case like 12,550. The first four significant figures in 0.6705067 are 6705.

18. Angles of Elevation and Depression. The *angle of elevation*, or the *angle of depression*, of an object is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane.



Thus, if the observer is at O, x is the angle of elevation of B, and y is the angle of depression of C.

In measuring angles with a transit the height of the instrument must always be taken into account. In stating problems, however, it is not convenient to consider this every time, and hence the angle is supposed to be taken from the level on which the instrument stands, unless otherwise stated.

1. From a point 5 ft. above the ground and 150 ft. from the foot of a tree the angle of elevation of the top is observed to be 20°. How high is the tree?

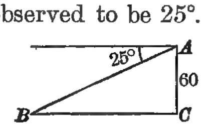
We have $a = b \tan A,$

or

$$\begin{aligned} a &= 150 \tan 20° \\ &= 150 \times 0.3640 \\ &= 54.6. \end{aligned}$$

Hence the height of the tree is 54.6 ft. + 5 ft., or 59.6 ft.

2. From a point A on a cliff 60 ft. high, including the instrument, the angle of depression of a boat B on a lake is observed to be 25°. How far is the boat from C, the foot of the cliff?



We have $\angle BAC = 65°$. Hence $BC = 60 \tan 65°$. From the table, $\tan 65° = 2.1445$. Hence $BC = 60 \times 2.1445 = 128.67$.

Exercise 8. Use of the Tangent

Find a to four significant figures, given the following:

1. $b = 37$, $A = 18°$.
2. $b = 26$, $A = 23°$.
3. $b = 48$, $A = 31°$.
4. $b = 62$, $A = 36°$.
5. $b = 98$, $A = 45°$.
6. $b = 4.8$, $A = 51°$.
7. $b = 9.6$, $A = 57°$.
8. $b = 23.4$, $A = 62°$.
9. $b = 28.7$, $A = 75°$.
10. $b = 39.7$, $A = 85°$.

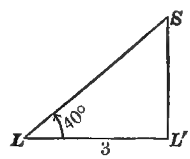
Find A, given the following:

11. $a = 6$, $b = 6$.
12. $a = 0.281$, $b = 2$.
13. $a = 4.752$, $b = 30$.
14. $a = 13.772$, $b = 40$.
15. $a = 2.424$, $b = 6$.
16. $a = 20.503$, $b = 10$.

17. A man standing 120 ft. from the foot of a church spire finds that the angle of elevation of the top is 50°. If his eye is 5 ft. 8 in. from the ground, what is the height of the spire?

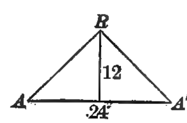
18. When a flagstaff 55.43 ft. high casts a shadow 100 ft. long on a horizontal plane, what is the angle of elevation of the sun?

19. A ship S is observed at the same instant from two lighthouses, L and L', 3 mi. apart. $\angle L'LS$ is found to be 40° and $\angle LL'S$ is found to be 90°. What is the distance of the ship from L'? What is its distance from L?



20. From the top of a rock which rises vertically, including the instrument, 134 ft. above a river bank the angle of depression of the opposite bank is found to be 40°. How wide is the river?

21. An A-shaped roof has a span AA' of 24 ft. The ridgepole R is 12 ft. above the horizontal line AA'. What angle does AR make with AA'? with RA'? with the perpendicular from R on AA'?



22. The foot of a ladder is 17 ft. 6 in. from a wall, and the ladder makes an angle of 42° with the horizontal when it leans against the wall. How far up the wall does it reach?

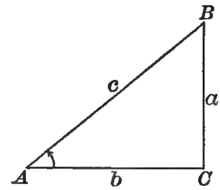
23. A post subtends an angle of 7° from a point on the ground 50 ft. away. What is the height of the post?

24. The diameter of a one-cent piece is $\frac{3}{4}$ in. If the coin is held so that it subtends an angle of 40° at the eye, what is its distance from the eye?

19. Practical Use of the Cotangent. Since by definition we have

$$\frac{b}{a} = \cot A,$$

we see that $$b = a \cot A.$$



For example, given $a = 71$ and $A = 28°$, find b.

From the table, $\cot 28° = 1.8807$
and $a = 71$

$$\begin{array}{r} 1\,8807 \\ 131\,649 \\ \hline 133.5297 \end{array}$$

Hence, to four significant figures, $b = 133.5$.

What check should be applied in this case?

Exercise 9. Use of the Cotangent

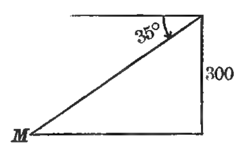
Find b to four significant figures, given the following:

1. $a = 29$, $A = 48°$.
2. $a = 38$, $A = 72°$.
3. $a = 56$, $A = 19°$.
4. $a = 72$, $A = 40°$.
5. $a = 425$, $A = 38°$.
6. $a = 19\frac{1}{2}$, $A = 36°$.
7. $a = 24.8$, $A = 43°$.
8. $a = 256.8$, $A = 75°$.

Find A, given the following:

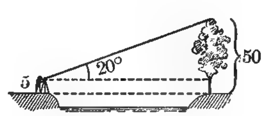
9. $a = 72$, $b = 72$.
10. $a = 60$, $b = 128.67$.

11. How far from a tree 50 ft. high must a person lie in order to see the top at an angle of elevation of 60°?



12. From the top of a tower 300 ft. high, including the instrument, a point on the ground is observed to have an angle of depression of 35°. How far is the point from the tower?

13. From the extremity of the shadow cast by a church spire 150 ft. high the angle of elevation of the top is 53°. What is the length of the shadow?



14. A tree known to be 50 ft. high, standing on the bank of a stream, is observed from the opposite bank to have an angle of elevation of 20°. The angle is measured on a line 5 ft. above the foot of the tree. How wide is the stream?

20. Practical Use of the Secant. Since by definition we have

$$\frac{c}{b} = \sec A,$$

we see that $$c = b \sec A.$$

For example, given $b = 15$ and $A = 30°$, find c.

From the table, $\sec 30° = 1.1547$

and $b = 15$

$$\begin{array}{r} 1.1547 \\ 15 \\ \hline 5\,7735 \\ 11\,547 \\ \hline 17.3205 \end{array}$$

Hence, to four significant figures, $c = 17.32$.

Exercise 10. Use of the Secant

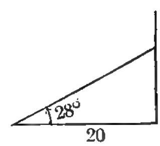
Find c to four significant figures, given the following:

1. $b = 36$, $A = 27°$.
2. $b = 48$, $A = 39°$.
3. $b = 74$, $A = 43°$.
4. $b = 22\frac{1}{2}$, $A = 48°$.
5. $b = 33.4$, $A = 53°$.
6. $b = 148.8$, $A = 64°$.

Find A, given the following:

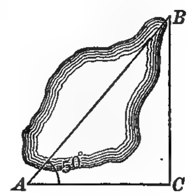
7. $b = 10$, $c = 13\frac{1}{4}$.
8. $b = 17.8$, $c = 35.6$.

9. A ladder rests against the side of a building, and makes an angle of 28° with the ground. The foot of the ladder is 20 ft. from the building. How long is the ladder?



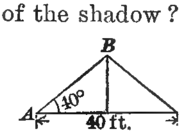
10. From a point 50 ft. from a house a wire is stretched to a window so as to make an angle of 30° with the horizontal. Find the length of the wire, assuming it to be straight.

11. In measuring the distance AB a surveyor ran the line AC, making an angle of 50° with AB, and the line BC perpendicular to AC. He measured AC and found that it was 880 ft. Required the distance AB.



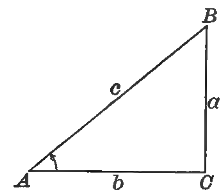
12. From the extremity of the shadow cast by a tree the angle of elevation of the top is 47°. The shadow is 62 ft. 6 in. long. How far is it from the top of the tree to the extremity of the shadow?

13. The span of this roof is 40 ft., and the roof timbers AB make an angle of 40° with the horizontal. Find the length of AB.



21. Practical Use of the Cosecant. Since by definition we have

$$\frac{c}{a} = \csc A,$$



we see that $$c = a \csc A.$$

For example, given $a = 22$ and $A = 35°$, find c.

From the table, $\csc 35° = 1.7434$
and $a = 22$

$$\begin{array}{r} 1.7434 \\ 22 \\ \hline 3\,4868 \\ 34\,868 \\ \hline 38.3548 \end{array}$$

Hence, to four significant figures, $c = 38.35$.

Check. Since $\frac{a}{c} = \sin A$, $22 \div 38.35 = 0.5736 = \sin 35°$.

Exercise 11. Use of the Cosecant

Find c to four significant figures, given the following:

1. $a = 24$, $A = 29°$.
2. $a = 36$, $A = 41°$.
3. $a = 56$, $A = 44°$.
4. $a = 56\frac{1}{2}$, $A = 61°$.
5. $a = 75.8$, $A = 69°$.
6. $a = 146.9$, $A = 74°$.

Find A, given the following:

7. $a = 10$, $c = 11.126$.
8. $a = 13$, $c = 27.6913$.
9. $a = 5\frac{1}{2}$, $c = 6.0687$.
10. $a = 75$, $c = 106.065$.

11. Seen from a point on the ground the angle of elevation of an aeroplane is 64°. If the aeroplane is 1000 ft. above the ground, how far is it in a straight line from the observer?

12. A ship sailing 47° east of north changes its latitude 28 mi. in 3 hr. What is its rate of sailing per hour?

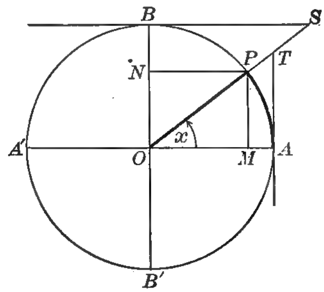
13. A ship sailing 63° east of south changes its latitude 45 mi. in 5 hr. What is its rate of sailing per hour?

14. From the top of a lighthouse 100 ft., including the instrument, above the level of the sea a boat is observed under an angle of depression of 22°. How far is the boat from the point of observation?

15. Seen from a point on the ground the angle of elevation of the top of a telegraph pole 27 ft. high is 28°. How far is it from the point of observation to the top of the pole?

16. What is the length of the hypotenuse of a right triangle of which one side is $11\frac{3}{4}$ in. and the opposite angle 43°?

22. Functions as Lines. The functions of an angle, being ratios, are *numbers*; but we may represent them by *lines* if we first choose a unit of length, and then construct right triangles, such that the denominators of the ratios shall be equal to this unit.



Thus in the annexed figure the radius is taken as 1, the circle then being spoken of as a *unit circle*. Then

$$OA = OP = OB = 1.$$

Drawing the four perpendiculars as shown, we have:

$$\sin x = \frac{MP}{OP} = MP; \qquad \cos x = \frac{OM}{OP} = OM;$$

$$\tan x = \frac{AT}{OA} = AT; \qquad \cot x = \frac{BS}{OB} = BS;$$

$$\sec x = \frac{OT}{OA} = OT; \qquad \csc x = \frac{OS}{OB} = OS.$$

In each case we have arranged the fraction so that the denominator is 1. For example, instead of taking $\frac{MP}{OM}$ for $\tan x$ we have taken the equal ratio $\frac{AT}{OA}$, because $OA = 1$.

Similarly, instead of taking $\frac{OP}{PM}$ for $\csc x$ we have taken the equal ratio $\frac{OS}{OB}$, because $OB = 1$.

This explains the use of the names *tangent* and *secant*, AT being a tangent to the circle, and OT being a secant.

Formerly the functions were considered as lines instead of ratios and received their names at that time. The word *sine* is from the Latin *sinus*, a translation of an Arabic term for this function.

We see from the figure that the sine of the complement of x is NP, which equals OM; also that the tangent of the complement of x is BS, and that the secant of the complement of x is OS.

Exercise 12. Functions as Lines

1. Represent by lines the functions of 45°.

2. Represent by lines the functions of an acute angle greater than 45°.

Using the above figure, determine which is the greater:

3. $\sin x$ or $\tan x$.	5. $\sec x$ or $\tan x$.	7. $\cos x$ or $\cot x$.
4. $\sin x$ or $\sec x$.	6. $\csc x$ or $\cot x$.	8. $\cos x$ or $\csc x$.

Construct the angle x, given the following:

9. $\tan x = 3$.
11. $\cos x = \frac{1}{2}$.
13. $\sin x = 2\cos x$
10. $\csc x = 2$.
12. $\sin x = \cos x$.
14. $4 \sin x = \tan x$.

15. Show that the sine of an angle is equal to one half the chord of twice the angle in a unit circle.

16. Find x if $\sin x$ is equal to one half the side of a regular decagon inscribed in a unit circle.

Given x and y, $x + y$ being less than $90°$; construct a line equal to

17. $\sin(x + y) - \sin x$.
20. $\cos x - \cos(x + y)$.
18. $\tan(x + y) - \tan x$.
21. $\cot x - \cot(x + y)$.
19. $\sec(x + y) - \sec x$.
22. $\csc x - \csc(x + y)$.
23. $\tan(x + y) - \sin(x + y) + \tan x - \sin x$.

Given an angle x, construct an angle y such that:

24. $\sin y = 2 \sin x$.
28. $\tan y = 3 \tan x$.
25. $\cos y = \frac{1}{2} \cos x$.
29. $\sec y = \csc x$.
26. $\sin y = \cos x$.
30. $\sin y = \frac{1}{2} \tan x$.
27. $\tan y = \cot x$.
31. $\sin y = \frac{2}{3} \tan x$.

32. Show by construction that $2 \sin A > \sin 2A$, when $A < 45°$.

33. Show by construction that $\cos A < 2 \cos 2A$, when $A < 45°$.

34. Given two angles A and B, $A + B$ being less than $90°$; show that $\sin(A + B) < \sin A + \sin B$.

35. Given $\sin x$ in a unit circle; find the length of a line in a circle of radius r corresponding in position to $\sin x$.

36. In a right triangle, given the hypotenuse c, and $\sin A = m$; find the two sides.

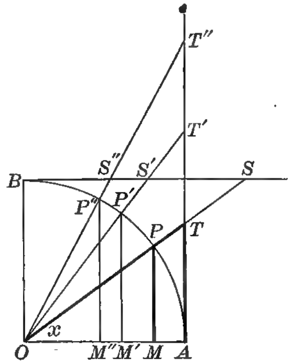
37. In a right triangle, given the side b, and $\tan A = m$; find the other side and the hypotenuse.

Construct, or show that it is impossible to construct, the angle x, given the following:

38. $\sin x = \frac{1}{2}$.
41. $\cos x = 0$.
44. $\tan x = \frac{4}{3}$.
39. $\sin x = 1$.
42. $\cos x = \frac{4}{3}$.
45. $\cot x = \frac{1}{2}$.
40. $\sin x = \frac{5}{4}$.
43. $\cos x = \frac{1}{3}$.
46. $\sec x = \frac{1}{2}$.

47. Using a protractor, draw the figure to show that $\sin 60° = \cos(\frac{1}{2} \text{ of } 60°)$, and $\sin 30° = \cos(2 \times 30°)$.

23. Changes in the Functions. If we suppose $\angle AOP$, or x, to increase gradually to 90°, the sine MP increases to $M'P'$, $M''P''$, and so on to OB.



That is, the sine increases from 0 for the angle 0°, to 1 for the angle 90°. Hence 0 and 1 are called the *limiting values* of the sine.

Similarly, AT and OT gradually increase in length, while OM, BS, and OS gradually decrease. That is,

As an acute angle increases to 90°, its sine, tangent, and secant also increase, while its cosine, cotangent, and cosecant decrease.

If we suppose x to decrease to 0°, OP coincides with OA and is parallel to BS. Therefore MP and AT vanish, OM becomes equal to OA, while BS and OS are each infinitely long and are represented in value by the symbol ∞. Similarly, we may consider the changes as x increases from 0° to 90°.

Hence, as the angle x increases from 0° to 90°, we see that

$\sin x$ increases from 0 to 1,
$\cos x$ decreases from 1 to 0,
$\tan x$ increases from 0 to ∞,
$\cot x$ decreases from ∞ to 0,
$\sec x$ increases from 1 to ∞,
$\csc x$ decreases from ∞ to 1.

We also see that

sines and cosines are never greater than 1;
secants and cosecants are never less than 1;
tangents and cotangents may have any values from 0 to ∞.

In particular, for the angle 0°, we have the following values:

$$\sin 0° = 0, \qquad \tan 0° = 0, \qquad \sec 0° = 1,$$
$$\cos 0° = 1, \qquad \cot 0° = \infty, \qquad \csc 0° = \infty.$$

For the angle 90° we have the following values:

$$\sin 90° = 1, \qquad \tan 90° = \infty, \qquad \sec 90° = \infty,$$
$$\cos 90° = 0, \qquad \cot 90° = 0, \qquad \csc 90° = 1.$$

By reference to the figure and the table it is apparent that the functions of 45° are never equal to half of the corresponding functions of 90°. Thus,

$$\sin 45° = 0.7071, \qquad \tan 45° = 1, \qquad \sec 45° = 1.4142,$$
$$\cos 45° = 0.7071, \qquad \cot 45° = 1, \qquad \csc 45° = 1.4142.$$

Exercise 13. Functions as Lines

1. Draw a figure to show that $\sin 90° = 1$.

2. What is the value of $\cos 90°$? Draw a figure to show this.

3. What is the value of $\sec 0°$? Draw a figure to show this.

4. What is the value of $\tan 90°$? Draw a figure to show this.

5. What is the value of $\cot 90°$? Draw a figure to show this.

6. As the angle increases, which increases the more rapidly, the sine or the tangent? Show this by reference to the figure.

7. If you double an angle, does this double the sine? Show this by reference to the figure.

8. If you bisect an angle, does this bisect the tangent? Prove it.

9. State the angle for which these relations are true:

$$\sin x = \cos x, \qquad \tan x = \cot x, \qquad \sec x = \csc x.$$

Show this by reference to the figure.

10. If you know that $\sin 40° 15' = 0.6461$, and $\cos 40° 15' = 0.7632$, and that the difference between each of these and the sine and cosine of $40° 15' 30''$ is 0.0001, what is $\sin 40° 15' 30''$? $\cos 40° 15' 30''$?

11. If you know that $\tan 20° 12'$ is 0.3679, and that the difference between this and $\tan 20° 12' 15''$ is 0.0001, what is $\tan 20° 12' 15''$?

12. If you know that $\cot 20° 12'$ is 2.7179, and that the difference between this and $\cot 20° 12' 15''$ is 0.0006, what is $\cot 20° 12' 15''$?

13. If you know that $\tan 66.5°$ is 2.2998, and that the difference between this and $\tan 66.6°$ is 0.0111, what is $\tan 66.6°$?

14. If you know that $\cos 57.4°$ is 0.5388, and that the difference between this and $\cos 57.5°$ is 0.0015, what is $\cos 57.5°$?

Draw the angle x for which the functions have the following values and state (page 11) to the nearest degree the value of the angle:

15. $\sin x = 0.1$.	21. $\tan x = 0.1$.	27. $\sec x = 1.2$.
16. $\sin x = 0.4$.	22. $\tan x = 0.23$.	28. $\sec x = 1.3$.
17. $\sin x = 0.7$.	23. $\tan x = 0.4$.	29. $\sec x = 1.7$.
18. $\cos x = 0.9$.	24. $\cot x = 4.0$.	30. $\csc x = 2.0$.
19. $\cos x = 0.8$.	25. $\cot x = 2.9$.	31. $\csc x = 3.6$.
20. $\cos x = 0.7$.	26. $\cot x = 0.9$.	32. $\csc x = 1.66$.

33. Find the value of $\sin x$ in the equation $\sin x - \dfrac{1}{\sin x} + 1.5 = 0$. Which root is admissible? Why is the other root impossible?

CHAPTER II

USE OF THE TABLE OF NATURAL FUNCTIONS

24. Sexagesimal and Decimal Fractions. The ancients, not having developed the idea of the decimal fraction and not having any convenient notation for even the common fraction, used a system based upon sixtieths. Thus they had units, sixtieths, thirty-six hundredths, and so on, and they used this system in all kinds of theoretical work requiring extensive fractions.

For example, instead of $1\frac{7}{15}$ they would use 1 28′, meaning $1\frac{28}{60}$; and instead of 1.51 they would use 1 30′ 36″, meaning $1\frac{30}{60}+\frac{36}{3600}$. The symbols for degrees, minutes, and seconds are modern.

We to-day apply these *sexagesimal* (scale of sixty) fractions only to the measure of time, angles, and arcs. Thus

$$3 \text{ hr. } 10 \text{ min. } 15 \text{ sec. means } (3+\tfrac{10}{60}+\tfrac{15}{3600}) \text{ hr.},$$

and $$3^\circ\, 10'\, 15'' \text{ means } (3+\tfrac{10}{60}+\tfrac{15}{3600})^\circ.$$

In medieval times the sexagesimal system was carried farther than this. For example, 3 10′ 20″ 30‴ 45^{iv} was used for $3+\frac{10}{60}+\frac{20}{60^2}+\frac{30}{60^3}+\frac{45}{60^4}$. Some writers used sexagesimal fractions in which the denominators extended to 60^{12}.

Since about the year 1600 we have had decimal fractions with which to work, and these have gradually replaced sexagesimal fractions in most cases. At present there is a strong tendency towards using decimal instead of sexagesimal fractions in angle measure. On this account it is necessary to be familiar with tables which give the functions of angles not only to degrees and minutes, but also to degrees and hundredths, with provision for finding the functions also to seconds and to thousandths of a degree. Hence the tables which will be considered and the problems which will be proposed will involve both sexagesimal and decimal fractions, but with particular attention to the former because they are the ones still commonly used.

The rise of the metric system in the nineteenth century gave an impetus to the movement to abandon the sexagesimal system. At the time the metric system was established in France, trigonometric tables were prepared on the decimal plan. It is only within recent years, however, that tables of this kind have begun to come into use.

25. Sexagesimal Table. The following is a portion of a page from the Wentworth-Smith Trigonometric Tables:

41°

′	sin	cos	tan	cot	′
0	6561	7547	8693	1.1504	60
1	6563	7545	8698	1.1497	59
2	6565	7543	8703	1.1490	58
3	6567	7541	8708	1.1483	57
4	6569	7539	8713	1.1477	56
5	6572	7538	8718	1.1470	55
...	...	...	...	...	...
′	cos	sin	cot	tan	′

48°

42°

′	sin	cos	tan	cot	′
0	6691	7431	9004	1.1106	60
1	6693	7430	9009	1.1100	59
2	6696	7428	9015	1.1093	58
3	6698	7426	9020	1.1087	57
4	6700	7424	9025	1.1080	56
5	6702	7422	9030	1.1074	55
...	...	...	...	...	...
′	cos	sin	cot	tan	′

47°

The functions of 41° and any number of minutes are found by reading down, under the abbreviations *sin, cos, tan, cot.*

For example, $\sin 41° = 0.6561$, $\sin 42° = 0.6691$,
$\cos 41°\, 2' = 0.7543$, $\cos 42° = 0.7431$,
$\tan 41°\, 4' = 0.8713$, $\tan 42°\, 3' = 0.9020$,
$\cot 41°\, 5' = 1.1470$, $\cot 42°\, 5' = 1.1074$.

Decimal points are usually omitted in the tables when it is obvious where they should be placed.

The secant and cosecant are seldom given in tables, being reciprocals of the cosine and sine. We shall presently see that we rarely need them.

Since $\sin 41°\, 2'$ is the same as $\cos 48°\, 58'$ (§ 8), we may use the same table for 48° and any number of minutes by reading up, above the abbreviations *cos, sin, cot, tan.*

For example, $\cos 48°\, 55' = 0.6572$, $\cos 47°\, 55' = 0.6702$,
$\sin 48°\, 56' = 0.7539$, $\sin 47°\, 56' = 0.7424$,
$\cot 48°\, 58' = 0.8703$, $\cot 47°\, 57' = 0.9020$,
$\tan 48°\, 59' = 1.1497$, $\tan 47°\, 59' = 1.1100$.

Trigonometric tables are generally arranged with the degrees from 0° to 44° at the top, the minutes being at the left; and with the degrees from 45° to 89° at the bottom, the minutes being at the right. Therefore, in looking for functions of an angle from 0° to 44° 59′, look at the top of the page for the degrees and in the left column for the minutes, reading the number below the proper abbreviation. For functions of an angle from 45° to 90° (89° 60′), look at the bottom of the page for the degrees and in the right-hand column for the minutes, reading the number above the proper abbreviation.

Exercise 14. Use of the Sexagesimal Table

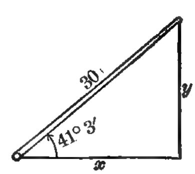
From the table on page 28 find the values of the following:

1. $\cos 41°$.	6. $\sin 48° 59'$.	11. $\sin 42° 4'$.
2. $\tan 42°$.	7. $\sin 47° 58'$.	12. $\cos 47° 56'$.
3. $\cos 41° 1'$.	8. $\cos 48° 59'$.	13. $\tan 41° 3'$.
4. $\tan 42° 2'$.	9. $\cos 47° 59'$.	14. $\cot 48° 57'$.
5. $\cos 41° 5'$.	10. $\cos 48° 57'$.	15. $\tan 48° 57'$.

In the right triangle ACB, in which $C = 90°$:

16. Given $c = 27$ and $A = 41° 3'$, find a.
17. Given $c = 48$ and $A = 42° 4'$, find a.
18. Given $c = 61$ and $A = 41° 2'$, find b.
19. Given $c = 72$ and $A = 42° 3'$, find b.
20. Given $b = 24$ and $A = 41° 3'$, find a.
21. Given $b = 28$ and $A = 42° 4'$, find a.
22. Given $a = 42$ and $A = 41° 1'$, find b.
23. Given $a = 60$ and $A = 42° 4'$, find b.
24. Given $c = 86$ and $A = 48° 56'$, find a.
25. Given $c = 92$ and $A = 48° 57'$, find a.
26. Given $b = 45$ and $A = 47° 55'$, find a.
27. Given $b = 85$ and $A = 47° 59'$, find a.
28. Given $a = 86$ and $A = 48° 56'$, find b.
29. Given $a = 98$ and $A = 47° 58'$, find b.
30. Given $b = 67$ and $c = 100$, find A.

31. A hoisting crane has an arm 30 ft. long. When the arm makes an angle of $41° 3'$ with x, what is the length of y? what is the length of x?



32. In Ex. 31 suppose the arm is raised until it makes an angle of $41° 5'$ with x, what are then the lengths of y and x?

33. From a point 128 ft. from a building the angle of elevation of the top is observed, by aid of an instrument 5 ft. above the ground, to be $42° 4'$. What is the height of the building?

34. From the top of a building 62 ft. 6 in. high, including the instrument, the angle of depression of the foot of an electric-light pole is observed to be $41° 3'$. How far is the pole from the building?

26. Decimal Table. It would be possible to have a decimal table of natural functions arranged as follows:

°	sin	cos	tan	cot	°
0.0	0000	1.0000	0000	∞	**90.0**
0.1	0017	1.0000	0017	573.0	89.9
0.2	0035	1.0000	0035	286.5	89.8
0.3	0052	1.0000	0052	191.0	89.7
0.4	0070	1.0000	0070	143.2	89.6
0.5	0087	1.0000	0087	114.6	**89.5**
...	...	...	...	...	...
°	cos	sin	cot	tan	°

°	sin	cos	tan	cot	°
4.0	0698	9976	0699	14.30	**86.0**
4.1	0715	9974	0717	13.95	85.9
4.2	0732	9973	0734	13.62	85.8
4.3	0750	9972	0752	13.30	85.7
4.4	0767	9971	0769	13.00	85.6
4.5	0785	9969	0787	12.71	**85.5**
...	...	...	...	...	...
°	cos	sin	cot	tan	°

Since, however, the decimal divisions of the angle have not yet become common, it is not necessary to have a special table of this kind. It is quite convenient to use the ordinary sexagesimal table for this purpose by simply referring to the Table of Conversion of sexagesimals to decimals and vice versa. This table is given with the other Wentworth-Smith tables prepared for use with this book. Thus if we wish to find $\sin 27.75°$, we see by the Table of Conversion that $0.75° = 45'$, so we simply look for $\sin 27° 45'$.

For example, using either the above table or, after conversion to sexagesimals, the common table, we see that:

$$\begin{aligned} \sin 0.4° &= 0.0070, & \sin 85.5° &\doteq 0.9969,\\ \cos 4.1° &= 0.9974, & \cos 85.5° &= 0.0785,\\ \tan 0.5° &= 0.0087, & \tan 85.8° &= 13.62,\\ \cot 4.3° &= 13.30, & \cot 85.9° &= 0.0717. \end{aligned}$$

Exercise 15. Use of the Decimal Table

From the above table find the values of the following:

1. $\sin 0.5°$.
2. $\tan 0.4°$.
3. $\sin 4°$.
4. $\cos 4.2°$.
5. $\tan 4.5°$.
6. $\sin 4.1°$.
7. $\cos 4.3°$.
8. $\tan 4.4°$.
9. $\cot 4.5°$.
10. $\cot 4.2°$.
11. $\sin 85.7°$.
12. $\sin 85.9°$.
13. $\cos 85.6°$.
14. $\tan 85.9°$.
15. $\cot 85.6°$.
16. $\sin 89.5°$.
17. $\cos 85.9°$.
18. $\tan 89.6°$.
19. $\cot 89.7°$.
20. $\cot 85.8°$.

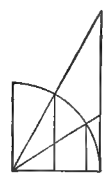
21. The hypotenuse of a right triangle is 12.7 in., and one acute angle is 85.5°. Find the two perpendicular sides.

22. From a point on the top of a house the angle of depression of the foot of a tree is observed to be 4.4°. The house, including the instrument, is 30 ft. high. How far is the tree from the house?

23. A rectangle has a base 9.5 in. long, and the diagonal makes an angle of 4.5° with the base. Find the height of the rectangle and the length of the diagonal.

27. Interpolation. So long as we wish to find the functions of an acute angle expressed in degrees and minutes, or in degrees and tenths, the tables already explained are sufficient. But when the angle is expressed in degrees, minutes, and seconds, or in degrees and hundredths, we see that the tables do not give the values of the functions directly. It is then necessary to resort to a process called *interpolation*.

Briefly expressed, in the process of interpolation we assume that $\sin 42\frac{1}{2}°$ is found by adding to $\sin 42°$ half the difference between $\sin 42°$ and $\sin 43°$.



In general it is evident that this is not true. For example, in the annexed figure the line values of the functions of 30° and 60° are shown. It is clear that $\sin 30°$ is more than half $\sin 60°$, that $\tan 30°$ is less than half $\tan 60°$, and that $\sec 30°$ is more than half $\sec 60°$. This is also seen from the table on page 11, where

$$\sin 30° = 0.5000, \qquad \tan 30° = 0.5774, \qquad \sec 30° = 1.1547,$$
$$\sin 60° = 0.8660, \qquad \tan 60° = 1.7321, \qquad \sec 60° = 2.0000.$$

For angles in which the changes are very small, interpolation gives results which are correct to the number of decimal places given in the table.

For example, from the table on page 11 we have

$$\sin 42° = 0.6691$$
$$\sin 41° = \underline{0.6561}$$

Difference for 1°, or 60′, $= 0.0130$

Difference for $1' = \frac{1}{60}$ of 0.0130 $= 0.0002.$

Adding this to sin 41°, we have

$$\sin 41°\ 1' = 0.6563,$$

a result given in the table on page 28.

But if we wish to find $\tan 89.6°$ from $\tan 89.5°$ and $\tan 89.7°$, we cannot use this method because here *the changes are very great*, as is always the case with the tangents and secants of angles near 90°, and with the cotangents and cosecants of angles near 0°. Thus, from the table on page 30,

$$\tan 89.7° = 191.0$$
$$\tan 89.5° = \underline{114.6}$$

Difference for 0.2° $= 76.4$

Difference for 0.1° $= 38.2$

Adding this to tan 89.5°, $\tan 89.6° = 152.8,$

whereas the table shows the result to be 143.2.

When cases arise in which interpolation cannot safely be used, we resort to the use of special tables that give the required values. These tables are explained later. Interpolation may safely be used in all examples given in the early part of the work.

28. Interpolation Applied. The following examples will illustrate the cases which arise in practical problems. The student should refer to the Wentworth-Smith Trigonometric Tables for the functions used in the problems.

1. Find $\sin 22° 10' 20''$.

From the tables, $\sin 22° 11' = 0.3776$

$\sin 22° 10' = 0.3773$

Difference for 1′, or 60″, the *tabular difference* $= 0.0003$

Difference for 20″ is $\frac{20}{60}$ of 0.0003, or 0.0001

Adding this to $\sin 22° 10'$, we have

$$\sin 22° 10' 20'' = 0.3774$$

2. Find $\cos 64° 17' 30''$.

From the tables, $\cos 64° 17' = 0.4339$

$\cos 64° 18' = 0.4337$

Tabular difference $= 0.0002$

Difference for 30″ is $\frac{30}{60}$ of 0.0002, or 0.0001

Since the cosine decreases as the angle increases we must subtract 0.0001 from $\cos 64° 17'$, which gives us

$$\cos 64° 17' 30'' = 0.4338$$

3. Find $\tan 37.54°$.

By the Table of Conversion, $0.54° = 32' 24''$.

From the tables, $\tan 37° 33' = 0.7687$

$\tan 37° 32' = 0.7683$

Tabular difference $= 0.0004$

Difference for 24″ is $\frac{24}{60}$, or 0.4, of 0.0004 $= 0.0002$

Adding this to $\tan 37° 32'$, we have

$$\tan 37.54° = \tan 37° 32' 24'' = 0.7685$$

4. Given $\sin x = 0.6456$, find x.

Looking in the tables for the sine that is a little less than 0.6456, and for the next larger sine, we have

$0.6457 = \sin 40° 13'$

$0.6455 = \sin 40° 12'$

$0.0002 =$ tabular difference

Therefore x lies between 40° 12′ and 40° 13′.

Furthermore, $0.6456 = \sin x$

$0.6455 = \sin 40° 12'$

$0.0001 =$ difference

But 0.0001 is $\frac{1}{2}$ of 0.0002, the tabular difference, so that x is halfway from 40° 12′ to 40° 13′. Therefore we add $\frac{1}{2}$ of 60″, or 30″, to 40° 12′.

Hence $x = 40° 12' 30''$.

We interpolate in a similar manner when we use a decimal table.

Exercise 16. Use of the Table

Find the values of the following:

1. $\sin 27° 10' 30''$.
2. $\sin 42° 15' 30''$.
3. $\sin 56° 29' 40''$.
4. $\sin 65° 29' 40''$.
5. $\cos 36° 14' 30''$.
6. $\cos 43° 12' 20''$.
7. $\cos 64° 18' 45''$.
8. $\tan 28° 32' 20''$.
9. $\tan 32° 41' 30''$.
10. $\tan 42° 38' 30''$.
11. $\tan 52° 10' 45''$.
12. $\tan 68° 12' 45''$.
13. $\tan 72° 15' 50''$.
14. $\tan 85° 17' 45''$.
15. $\tan 86° 15' 50''$.
16. $\cot 5° 27' 30''$.
17. $\cot 6° 32' 45''$.
18. $\cot 7° 52' 50''$.
19. $\cot 8° 40' 10''$.
20. $\cot 9° 20' 10''$.

21. Given $\sin x = 0.6391$, find x. Then find $\cos x$.
22. Given $\sin x = 0.7691$, find x. Then find $\cos x$.
23. Given $\cos x = 0.3174$, find x. Then find $\sin x$.
24. Given $\tan x = 2.8649$, find x. Then find $\cot x$.
25. Given $\tan x = 5.3977$, find x. Then find $\cot x$.

First converting to sexagesimals, find the following:

26. $\sin 25.5°$.
27. $\sin 25.55°$.
28. $\sin 32.75°$.
29. $\sin 41.65°$.
30. $\sin 64.75°$.
31. $\cos 78.52°$.
32. $\tan 78.59°$.
33. $\cos 81.43°$.
34. $\tan 82.72°$.
35. $\tan 84.68°$.
36. $\cos 11.25°$.
37. $\cot 12.32°$.
38. $\cot 13.54°$.
39. $\cot 15.48°$.
40. $\cot 16.62°$.

Find the value of x in each of the following equations:

41. $\sin x = 0.5225$.
42. $\sin x = 0.5771$.
43. $\sin x = 0.6601$.
44. $\sin x = 0.7023$.
45. $\cos x = 0.7853$.
46. $\cos x = 0.7716$.
47. $\cos x = 0.9524$.
48. $\cos x = 0.7115$.
49. $\tan x = 2.6395$.
50. $\tan x = 4.7625$.
51. $\tan x = 4.7608$.
52. $\cot x = 3.7983$.

53. If $\sin x = 0.6431$, what is the value of $\cos x$?
54. If $\cos x = 0.7652$, what is the value of $\sin x$?
55. If $\tan x = 0.6827$, what is the value of $\sin x$?
56. If $\tan x = 0.6537$, what is the value of x? of $\cot x$?
57. If $\cot x = 1.6550$, what is the value of x? of $\tan x$? Verify the second result by the relation $\tan x = 1/\cot x$.

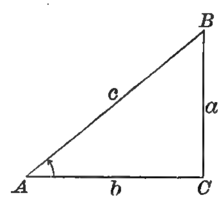
29. Application to the Right Triangle. In §§ 15–21 we learned how to use the several functions in finding various parts of a right triangle from other given parts, the angles being in exact degrees. In §§ 25–28 we learned how to use the tables when the angles were not necessarily in exact degrees. We shall now review both of these phases of the work in connection with the solution of the right triangle.

In order to *solve* a right triangle, that is, to find both of the acute angles, the hypotenuse, and both of the sides, two independent parts besides the right angle must be given.

In speaking of the *sides* of a right triangle it should be repeated that we shall refer only to sides a and b, the sides which include the right angle, using the word *hypotenuse* to refer to c. It will be found that there is no confusion in thus referring to only two of the three sides by the special name *sides*.

By *independent parts* is meant parts that do not depend one upon another. For example, the two acute angles are not independent parts, for each is equal to 90° minus the other.

The two given parts may be:



1. An acute angle and the hypotenuse.

That is, given A and c, or B and c. If A and c are given, we have to find a and b. The angle B is known from the relation $B = 90° - A$. If B is given, we can find A from the equation $A = 90° - B$.

2. An acute angle and the opposite side.

That is, given A and a, or B and b. If A and a are given, we have to find B, b, and c, and similarly for the other case.

3. An acute angle and the adjacent side.

That is, given A and b, or B and a. If A and b are given, we have to find B, a, and c, and similarly for the other case.

4. The hypotenuse and a side.

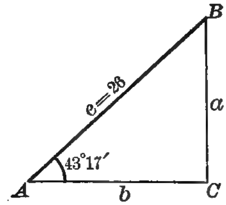
That is, given c and a, or c and b. If c and a are given, we have to find A, B, and b, and similarly for the other case.

5. The two sides.

That is, given a and b, to find A, B, and c. Using *side* to include hypotenuse, we might combine the fourth and fifth of these cases in one.

In each of these cases we shall consider right triangles which have their acute angles expressed in degrees and minutes, in degrees, minutes, and seconds, or in degrees and decimal parts of a degree. In this chapter the angles are given and required only to the nearest minute.

30. Given an Acute Angle and the Hypotenuse. For example, given $A = 43° 17'$, $c = 26$, find B, a, and b.



1. $B = 90° - A = 46° 43'$.

2. $\dfrac{a}{c} = \sin A$; $\therefore a = c \sin A$.

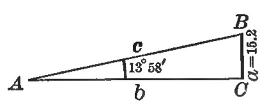
3. $\dfrac{b}{c} = \cos A$; $\therefore b = c \cos A$.

$$\begin{array}{rr}
\sin A = & 0.6856 \\
c = & 26 \\ \hline
 & 4\,1136 \\
 & 13\,712 \\ \hline
a = & 17.8256 \\
= & 17.83
\end{array}
\qquad\qquad
\begin{array}{rr}
\cos A = & 0.7280 \\
c = & 26 \\ \hline
 & 4\,3680 \\
 & 14\,560 \\ \hline
b = & 18.9280 \\
= & 18.93
\end{array}$$

As usual, when a four-place table is employed, the result is given to four figures only. The check is left for the student.

31. Given an Acute Angle and the Opposite Side. For example, given $A = 13° 58'$, $a = 15.2$, find B, b, and c.

1. $B = 90° - A = 76° 2'$.



2. $\dfrac{b}{a} = \cot A$; $\therefore b = a \cot A$.

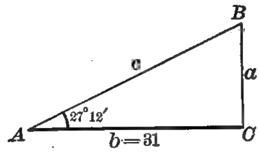
3. $\dfrac{a}{c} = \sin A$; $\therefore c = \dfrac{a}{\sin A}$.

$$\begin{array}{rr}
\multicolumn{2}{l}{a = 15.2,\ \cot A = 4.0207} \\
 & 4.0207 \\
 & 15.2 \\ \hline
 & 80414 \\
 & 20\,1035 \\
 & 40\,207 \\ \hline
b = & 61.11464 \\
= & 61.11
\end{array}
\qquad\qquad
\begin{array}{l}
a = 15.2,\ \sin A = 0.2414 \\
\qquad\qquad 62.97 = c \\
2414\overline{)152000.00} \\
\qquad 14484 \\ \hline
\qquad\ \ 7160 \\
\qquad\ \ 4828 \\ \hline
\qquad\ \ 23320 \\
\qquad\ \ 21726 \\ \hline
\end{array}$$

In dividing 15.2 by 0.2414, we adopt the modern plan of first multiplying each by 10,000. Only part of the actual division is shown.

Instead of dividing a by $\sin A$ to find c, we might multiply a by $\csc A$, as on page 22, except that tables do not generally give the cosecants. It will be seen in Chapter III that, by the aid of logarithms, we can divide by $\sin A$ as readily as multiply by $\csc A$, and this is why the tables omit the cosecant.

32. Given an Acute Angle and the Adjacent Side. For example, given $A = 27° 12'$, $b = 31$, find B, a, and c.



1. $B = 90° - A = 62° 48'$.

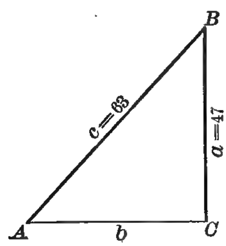
2. $\frac{a}{b} = \tan A$; $\therefore a = b \tan A$.

3. $\frac{b}{c} = \cos A$; $\therefore c = \frac{b}{\cos A}$.

$$\begin{array}{rr} \tan A = & 0.5139 \\ b = & \underline{31} \\ & 5139 \\ & \underline{15\,417} \\ a = & 15.9309 \\ & = 15.93 \end{array} \qquad \begin{array}{r} b = 31,\ \cos A = 0.8894 \\ 34.85 = c \\ 8894)\overline{310000.00} \\ \underline{26682} \\ 43180 \\ \underline{35576} \end{array}$$

We might multiply b by $\sec A$ instead of dividing by $\cos A$. The reason for not doing so is the same as that given in § 31 for not multiplying by $\csc A$.

33. Given the Hypotenuse and a Side. For example, given $a = 47$, $c = 63$, find A, B, and b.



1. $\sin A = \frac{a}{c}$.

2. $B = 90° - A$.

3. $b = \sqrt{c^2 - a^2}$
   $= \sqrt{(c + a)(c - a)}$.

In the case of $\sqrt{c^2 - a^2}$ we can, of course, square c, square a, take the difference of these squares, and then extract the square root. It is, however, easier to proceed by factoring $c^2 - a^2$ as shown. This will be even more apparent when we come, in Chapter III, to the short methods of computing by logarithms.

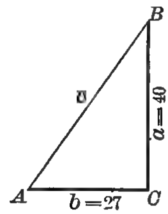
$$\begin{array}{lr} a = 47,\ c = 63 & \\ & 0.7460 \\ & 63)\overline{47.0000} \\ & \underline{44\,1} \\ & 2\,90 \\ \sin A = 0.7460 & \underline{2\,52} \\ \therefore A = 48° 15' & 380 \\ \therefore B = 41° 45' & \underline{378} \end{array} \qquad \begin{array}{rr} c + a = & 110 \\ c - a = & \underline{16} \\ & 660 \\ & \underline{110} \\ c^2 - a^2 = & 1760 \\ \therefore b^2 = & 1760 \\ \therefore b = & \sqrt{1760} \\ & = 41.95 \end{array}$$

34. Given the two Sides. For example, given $a = 40$, $b = 27$, find A, B, and c.

1. $\tan A = \dfrac{a}{b}$.

2. $B = 90° - A$.

3. $c = \sqrt{a^2 + b^2}$.



Of course c can be found in other ways. For example, after finding $\tan A$ we can find A, and hence can find $\sin A$. Then, because $\sin A = a/c$, we have $c = a/\sin A$. When the numbers are small, however, it is easy to find c from the relation given above.

$$\begin{aligned} a &= 40,\ b = 27 \\ \tfrac{40}{27} &= 1.4815 \\ \tan A &= 1.4815 \\ \therefore A &= 55°\ 59' \\ \therefore B &= 34°\ 1' \end{aligned} \qquad \begin{aligned} a^2 &= 1600 \\ b^2 &= \underline{\ 729} \\ c^2 &= 2329 \\ \therefore c &= \sqrt{2329} \\ &= 48.26 \end{aligned}$$

35. Checks. As already stated, always apply some check to the results. For example, in § 34, we see at once that $a^2 = 1600$ and b^2 is less than 30^2, or 900, so that c^2 is less than 2500, and c is less than 50. Hence the result as given, 48.26, is probably correct.

We can also find B independently.

For since $\tan B = \dfrac{b}{a}$,

we see that $\tan B = \frac{27}{40} = 0.6750$,

and therefore that $B = 34°\ 1'$.

Exercise 17. The Right Triangle

Solve the right triangle ACB, in which $C = 90°$, given:

1. $a = 3$, $b = 4$.
2. $a = 7$, $c = 13$.
3. $a = 5.3$, $A = 12°\ 17'$.
4. $a = 10.4$, $B = 43°\ 18'$.
5. $c = 26$, $A = 37°\ 42'$.
6. $c = 140$, $B = 24°\ 12'$.
7. $b = 19$, $c = 23$.
8. $b = 98$, $c = 135.2$.
9. $b = 42.4$, $A = 32°\ 14'$.
10. $b = 200$, $B = 46°\ 11'$.
11. $a = 95$, $b = 37$.
12. $a = 6$, $c = 103$.
13. $a = 3.12$, $B = 5°\ 8'$.
14. $a = 17$, $c = 18$.
15. $c = 57$, $A = 38°\ 29'$.
16. $a + c = 18$, $b = 12$.
17. $a + c = 90$, $b = 30$.
18. $a + c = 45$, $b = 30$.

Solve the right triangle ACB, in which $C = 90°$, given:

19. $a = 2.5$, $A = 35° 10' 30''$.
20. $a = 5.7$, $A = 42° 12' 30''$.
21. $a = 6.4$, $B = 29° 18' 30''$.
22. $a = 7.9$, $B = 36° 20' 30''$.
23. $c = 6.8$, $A = 29° 42' 30''$.
24. $c = 360$, $A = 34° 20' 30''$.
25. $b = 250$, $A = 41° 10' 40''$.
26. $a = 48$, $A = 25.5°$.
27. $c = 25$, $A = 24.5°$.
28. $c = 40$, $A = 32.55°$.
29. $c = 80$, $A = 55.51°$.
30. $c = 75$, $A = 63.46°$.
31. $a = 45$, $B = 50.59°$.
32. $b = 90$, $A = 68.25°$.

33. Each equal side of an isosceles triangle is 16 in., and one of the equal angles is 24° 10′. What is the length of the base?

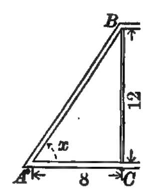
34. Each equal side of an isosceles triangle is 25 in., and the vertical angle is 36° 40′. What is the altitude of the triangle?

35. Each equal side of an isosceles triangle is 25 in., and one of the equal angles is 32° 20′ 30″. What is the length of the base?

36. Each equal side of an isosceles triangle is 60 in., and the vertical angle is 50° 30′ 30″. What is the altitude of the triangle?

37. Find the altitude of an equilateral triangle of which the side is 50 in. Show three methods of finding the altitude.

38. What is the side of an equilateral triangle of which the altitude is 52 in.?



39. In planning a truss for a bridge it is necessary to have the upright $BC = 12$ ft., and the horizontal $AC = 8$ ft., as shown in the figure. What angle does AB make with AC? with BC?

40. In Ex. 39 what are the angles if $AB = 12$ ft. and $AC = 9$ ft.?

41. In the figure of Ex. 39, what is the length of BC if $AB = 15$ ft. and $x = 62° 10'$?

42. Two angles of a triangle are 42° 17′ and 47° 43′ respectively, and the included side is 25 in. Find the other two sides.

43. A tangent AB, drawn from a point A to a circle, makes an angle of 51° 10′ with a line from A through the center. If $AB = 10$ ft., what is the length of the radius?

44. How far from the center of a circle of radius 12 in. will a tangent meet a diameter with which it makes an angle of 10° 20′?

45. Two circles of radii 10 in. and 14 in. are externally tangent. What angle does their line of centers make with their common exterior tangent?

CHAPTER III

LOGARITHMS

36. Importance of Logarithms. It has already been seen that the trigonometric functions are, in general, incommensurable with unity. Hence they contain decimal fractions of an infinite number of places. Even if we express these fractions only to four or five decimal places, the labor of multiplying and dividing by them is considerable. For this reason numerous devices have appeared for simplifying this work. Among these devices are various calculating machines, but none of these can easily be carried about and they are too expensive for general use. There is also the slide rule, an inexpensive instrument for approximate multiplication and division, but for trigonometric work this is not of particular value because the tables must be at hand even when the slide rule is used. The most practical device for the purpose was invented early in the seventeenth century and the credit is chiefly due to John Napier, a Scotchman, whose tables appeared in 1614. These tables, afterwards much improved by Henry Briggs, a contemporary of Napier, are known as tables of logarithms, and by their use the operation of multiplication is reduced to that of addition; that of division is reduced to subtraction; raising to any power is reduced to one multiplication; and the extracting of any root is reduced to a single division.

For the ordinary purposes of trigonometry the tables of functions used in Chapter II are fairly satisfactory, the time required for most of the operations not being unreasonable. But when a problem is met which requires a large amount of computation, the tables of natural functions, as they are called, to distinguish them from the tables of logarithmic functions, are not convenient.

For example, we shall see that the product of 2.417, 3.426, 517.4, and 91.63 can be found from a table by adding four numbers which the table gives.

In the case of $\frac{4.27}{52.9} \times \frac{36.1}{5.28} \times \frac{5176}{9283}$ we shall see that the result can be found from a table by adding six numbers.

Taking a more difficult case, like that of $\sqrt[3]{\frac{523}{711} \times \frac{9.64}{0.379}}$, we shall see that it is necessary merely to take one third of the sum of four numbers, after which the table gives us the result.

37. Logarithm. The power to which a given number, called the *base,* must be raised in order to be equal to another given number is called the *logarithm* of this second given number.

For example, since $10^2 = 100,$

we have, to the base 10, $2 =$ the logarithm of 100.

In the same way, since $10^3 = 1000,$

we have, to the base 10, $3 =$ the logarithm of 1000.

Similarly, $4 =$ the logarithm of 10,000,

$5 =$ the logarithm of 100,000,

and so on, whatever powers of 10 we take.

In general, if $b^x = N,$

then, to the base b, $x =$ the logarithm of N.

38. Symbolism. For "logarithm of N" it is customary to write "$\log N$." If we wish to specify $\log N$ to the base b, we write $\log_b N$, reading this "logarithm of N to the base b."

That is, as above, $\log 100 = 2,\quad \log 10{,}000 = 4,$

$\log 1000 = 3,\quad \log 100{,}000 = 5,$

and so on for the other powers of 10.

39. Base. Any positive rational number may be taken as the base for a system of logarithms, but 10 is usually taken for purposes of practical calculation.

Thus, since $2^3 = 8,\quad \log_2 8 = 3;$

since $3^4 = 81,\quad \log_3 81 = 4;$

and since $5^4 = 625,\quad \log_5 625 = 4.$

It is more convenient to take 10 as the base, however. For since

$$10^2 = 100 \text{ and } 10^3 = 1000,$$

we can tell at once that the logarithm of any number between 100 and 1000 must lie between 2 and 3, and therefore must be 2 + some fraction. That is, by using 10 as the base we know immediately the integral part of the logarithm.

When we write $\log 27$, we mean $\log_{10} 27$; that is, the base 10 is to be understood unless some other base is specified.

Since $\log 10 = 1,$ because $10^1 = 10,$

and $\log 1 = 0,$ because $10^0 = 1,$

and $\log \frac{1}{10} = -1,$ because $10^{-1} = \frac{1}{10},$

we see that *the logarithm of the base is always 1, the logarithm of 1 is always zero, and the logarithm of a proper fraction is negative.*

That this is true for any base is apparent from the fact that

$$b^1 = b, \quad \text{whence} \quad \log_b b = 1;$$

$$b^0 = 1, \quad \text{whence} \quad \log_b 1 = 0;$$

$$b^{-n} = \frac{1}{b^n}, \quad \text{whence} \quad \log_b \frac{1}{b^n} = -n.$$

Exercise 18. Logarithms

1. Since $2^5 = 32$, what is $\log_2 32$?
2. Since $4^2 = 16$, what is $\log_4 16$?
3. Since $10^4 = 10{,}000$, what is $\log 10{,}000$?

Write the following logarithms:

4. $\log_2 16$.
5. $\log_2 64$.
6. $\log_2 128$.
7. $\log_2 256$.
8. $\log_3 243$.
9. $\log_3 729$.
10. $\log_4 256$.
11. $\log_5 125$.
12. $\log_6 36$.
13. $\log_7 343$.
14. $\log_8 512$.
15. $\log_9 6561$.
16. $\log 100$.
17. $\log 1000$.
18. $\log 100{,}000$.
19. $\log 1{,}000{,}000$.

20. Since $10^{-1} = \frac{1}{10}$, or 0.1, what is $\log 0.1$?

21. What is $\log \frac{1}{100}$, or $\log 0.01$? $\log 0.001$? $\log 0.0001$?

22. Between what consecutive integers is $\log 52$? $\log 726$? $\log 2400$? $\log 24{,}000$? $\log 175{,}000$? $\log 175{,}000{,}000$?

23. Between what consecutive negative integers is $\log 0.08$? $\log 0.008$? $\log 0.0008$? $\log 0.1238$? $\log 0.0123$? $\log 0.002768$?

24. To the base 2, write the logarithms of 2, 4, 8, 64, 512, 1024, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$.

25. To the base 3, write the logarithms of 3, 81, 729, 2187, 6561, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$, $\frac{1}{729}$, $\frac{1}{2187}$.

26. To the base 10, write the logarithms of 1, 0.0001, 0.00001, 10,000,000, 100,000,000.

Write the consecutive integers between which the logarithms of the following numbers lie:

27. 75.
28. 75.9.
29. 75.05.
30. 82.95.
31. 642.
32. 642.75.
33. 642.005.
34. 793.175.
35. 7346.
36. 7346.9.
37. 7346.09.
38. 9182.735.
39. 243,481.
40. 5,276,192.
41. 7,286,348.5.
42. 19,423,076.

Show that the following statements are true:

43. $\log_2 4 + \log_2 8 + \log_2 16 + \log_2 64 + \log_2 2 + \log_2 32 = 21$.
44. $\log_3 3 + \log_3 9 + \log_3 81 + \log_3 729 + \log_3 27 + \log_3 243 = 21$.
45. $\log_{11} 11 + \log_{11} 121 + \log_{11} 1331 + \log_{11} 14{,}641 = 10$.
46. $\log 1 + \log 10 + \log 1000 + \log 0.1 + \log 0.001 = 0$.
47. $\log 1 + \log 100 + \log 10{,}000 + \log 0.01 + \log 0.0001 = 0$.
48. $\log 10{,}000 - \log 1000 + \log 100{,}000 - \log 100 = 4$.

40. Logarithm of a Product. *The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.*

Let A and B be the numbers, and x and y their logarithms. Then, taking 10 as the base and remembering that $x = \log A$, and $y = \log B$, we have

$$A = 10^x,$$

and

$$B = 10^y.$$

Therefore

$$AB = 10^{x+y},$$

and therefore

$$\log AB = x + y$$
$$= \log A + \log B.$$

The proof is the same if any other base is taken. For example,

if $x = \log_b A$, we have $A = b^x$;

and if $y = \log_b B$, we have $B = b^y$.

Therefore $AB = b^{x+y}$,

and

$$\log_b AB = x + y$$
$$= \log_b A + \log_b B.$$

The proposition is also true for the product of more than two numbers, the proof being evidently the same. Thus,

$$\log ABC = \log A + \log B + \log C,$$

and so on for any number of factors.

41. Logarithm of a Quotient. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For if

$$A = 10^x,$$

and

$$B = 10^y,$$

then

$$\frac{A}{B} = 10^{x-y},$$

and therefore

$$\log \frac{A}{B} = x - y$$
$$= \log A - \log B.$$

This proposition is true if any base b is taken. For, as in § 40,

$$\frac{A}{B} = b^{x-y},$$

and therefore

$$\log_b \frac{A}{B} = x - y$$
$$= \log_b A - \log_b B.$$

It is therefore seen from §§ 40 and 41 that if we know the logarithms of all numbers we can find the logarithm of a product by addition and the logarithm of a quotient by subtraction. If we can then find the numbers of which these results are the logarithms, we shall have solved our problems in multiplication and division by merely adding and subtracting.

42. Logarithm of a Power. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent.*

For if $A = 10^x$,

raising to the pth power, $A^p = 10^{px}$.

Hence
$$\log A^p = px$$
$$= p \log A.$$

This is easily seen by taking special numbers. Thus if we take the base 2, we have the following relations:

Since $2^5 = 32$, then $\log_2 32 = 5$;

and since $(2^5)^2 = 32^2 = 1024$, then $\log_2 1024 = 2 \cdot 5$
$$= 2 \log_2 32.$$

That is, $\log_2 32^2 = 2 \log_2 32$.

43. Logarithm of a Root. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

For if $A = 10^x$,

taking the rth root, $A^{\frac{1}{r}} = 10^{\frac{x}{r}}$.

Hence
$$\log A^{\frac{1}{r}} = \frac{x}{r}$$
$$= \frac{\log A}{r}.$$

The propositions of §§ 42 and 43 are true whatever base is taken, as may easily be seen by using the base b.

From §§ 42 and 43 we see that the raising of a number to any power, integral or fractional, reduces to the operation of multiplying the logarithm by the exponent (integral or fractional) and then finding the number of which the result is the logarithm.

Therefore the operations of multiplying, dividing, raising to powers, and extracting roots will be greatly simplified if we can find the logarithms of numbers, and this will next be considered.

44. Characteristic and Mantissa. Usually a logarithm consists of an integer plus a decimal fraction.

The integral part of a logarithm is called the *characteristic.*

The decimal part of a logarithm is called the *mantissa.*

Thus, if $\log 2353 = 3.37162$, the characteristic is 3 and the mantissa 0.37162. This means that $10^{3.37162} = 2353$, or that the 100,000th root of the 337,162d power of 10 is 2353, approximately.

It must always be recognized that the mantissa is only an approximation, correct to as many decimal places as are given in the table, but not exact. Computations made with logarithms give results which, in general, are correct only to a certain number of figures, but results which are sufficiently near the correct result to answer the purposes of the problem.

45. Finding the Characteristic. Since we know that

$$10^3 = 1000 \qquad \text{and} \qquad 10^4 = 10{,}000,$$

therefore

$$3 = \log 1000 \quad \text{and} \quad 4 = \log 10{,}000.$$

Hence the logarithm of a number between 1000 and 10,000 lies between 3 and 4, and is therefore 3 plus some fraction. Therefore the characteristic of a number between 1000 and 10,000 is 3.

Likewise, since

$$10^{-3} = 0.001 \qquad \text{and} \quad 10^{-2} = 0.01,$$

therefore

$$-3 = \log 0.001 \quad \text{and} \quad -2 = \log 0.01.$$

Therefore the logarithm of a number between 0.001 and 0.01 lies between -3 and -2, and hence is -3 plus some fraction. Therefore the characteristic of a number between 0.01 and 0.001 is -3.

Of course, instead of saying that $\log 1475$ is $3 +$ a fraction, we might say that it is $4 -$ a fraction; and instead of saying that $\log 0.007$ is $-3 +$ a fraction, we might say that it is $-2 -$ a fraction. For convenience, however, *the mantissa of a logarithm is always taken as positive*, but the characteristic may be either positive or negative.

46. Laws of the Characteristic. From the reasoning set forth in § 45 we deduce the following laws:

1. *The characteristic of the logarithm of a number greater than 1 is positive and is one less than the number of integral places in the number.*

For example,
$$\log 75 = 1 + \text{some mantissa},$$
$$\log 472.8 = 2 + \text{some mantissa},$$
and
$$\log 14{,}800.75 = 4 + \text{some mantissa}.$$

2. *The characteristic of the logarithm of a number between 0 and 1 is negative and is one greater than the number of zeros between the decimal point and the first significant figure in the number.*

For example,
$$\log 0.02 = -2 + \text{some mantissa},$$
and
$$\log 0.00076 = -4 + \text{some mantissa}.$$

The logarithm of a negative number is an imaginary number, and hence such logarithms are not used in computation.

47. Negative Characteristic. If $\log 0.02 = -2 + 0.30103$, we cannot write it -2.30103, because this would mean that both mantissa and characteristic are negative. Hence the form $\bar{2}.30103$ has been chosen, which means that only the characteristic 2 is negative.

That is, $\bar{2}.30103 = -2 + 0.30103$, and $\bar{5}.48561 = -5 + 0.48561$. We may also write $\bar{2}.30103$ as $0.30103 - 2$, or $8.30103 - 10$, or in any similar manner which will show that the characteristic is negative.

48. Mantissa independent of Decimal Point. It may be shown that

$$10^{3.37107} = 2350\text{; whence } \log 2350 = 3.37107.$$

Dividing 2350 by 10, we have

$$10^{3.37107-1} = 10^{2.37107} = 235\text{; whence } \log 235 = 2.37107.$$

Dividing 2350 by 10^4, or 10,000, we have

$$10^{3.37107-4} = 10^{\bar{1}.37107} = 0.235\text{; whence } \log 0.235 = \bar{1}.37107.$$

That is, the mantissas are the same for log 2350, log 235, log 0.235, and so on, wherever the decimal points are placed.

The mantissa of the logarithm of a number is unchanged by any change in the position of the decimal point of the number.

This is a fact of great importance, for if the table gives us the mantissa of log 235, we know that we may use the same mantissa for log 0.00235, log 2.35, log 23,500, log 235,000,000, and so on.

Exercise 19. Logarithms

Write the characteristics of the logarithms of the following:

1. 75.	6. 2578.	11. 0.8.	16. 0.0007.
2. 75.4.	7. 257.8.	12. 0.08.	17. 0.0077.
3. 754.	8. 25.78.	13. 0.88.	18. 0.00007.
4. 7.54.	9. 2.578.	14. 0.885.	19. 0.10007.
5. 7540.	10. 25,780.	15. 0.005.	20. 0.07007.

Given 3.58681 as the logarithm of 3862, find the following:

21. log 38.62.	24. log 38,620.	27. log 0.3862.
22. log 3.862.	25. log 386,200.	28. log 0.03862.
23. log 386.2.	26. log 38,620,000.	29. log 0.0003862.

Given $\bar{1}.67724$ as the logarithm of 0.4756, find the following:

30. log 4756.	32. log 47,560.	34. log 0.04756.
31. log 4.756.	33. log 47,560,000.	35. log 0.00004756

Given 3.40603 as the logarithm of 2547, find the following:

36. log 2.547.	38. log 0.2547.	40. log 25,470.
37. log 25.47.	39. log 0.002547.	41. log 25,470,000.

Given 1.39794 as the logarithm of 25, find the following:

42. log $2\frac{1}{2}$.	44. log 0.25.	46. log 25,000.
43. log $\frac{1}{4}$.	45. log 0.025.	47. log 25,000,000.

49. Using the Table. The following is a portion of a page taken from the Wentworth-Smith Logarithmic and Trigonometric Tables:

250 — 300

N	0	1	2	3	4	5	6	7	8	9
250	39 794	39 811	39 829	39 846	39 863	39 881	39 898	39 915	39 933	39 950
251	39 967	39 98$\underline{5}$	40 002	40 019	40 037	40 054	40 071	40 088	40 106	40 123
252	40 140	40 157	40 17$\underline{5}$	40 192	40 209	40 226	40 243	40 261	40 278	40 29$\underline{5}$
253	40 312	40 329	40 346	40 364	40 381	40 398	40 41$\underline{5}$	40 432	40 449	40 466
254	40 483	40 500	40 518	40 53$\underline{5}$	40 552	40 569	40 586	40 603	40 620	40 637
255	40 654	40 671	40 688	40 705	40 722	40 739	40 756	40 773	40 790	40 807

Only the mantissas are given; the characteristics are always to be determined by the laws stated in § 46. *Always write the characteristic at once, before writing the mantissa.*

For example, looking to the right of 251 and under 0, and writing the proper characteristics, we have

$$\log 251 = 2.39967, \qquad \log 25.1 = 1.39967,$$
$$\log 2510 = 3.39967, \qquad \log 0.0251 = \bar{2}.39967.$$

The first three significant figures of each number are given under **N**, and the fourth figure under the columns headed 0, 1, 2, . . ., 9.

For example, $\log 252.1 = 2.40157, \qquad \log 0.2547 = \bar{1}.40603,$
$$\log 25.25 = 1.40226, \qquad \log 2549 = 3.40637.$$

Furthermore, $\log 251.1 = 2.39985-$, the minus sign being placed beneath the final 5 in the table to show that if only a four-place mantissa is being used it should be written 3998 instead of 3999.

The logarithms of numbers of more than four figures are found by interpolation, as explained in § 27.

For example, to find log 25,314 we have

$$\begin{array}{rl} \log 25{,}320 = & 4.40346 \\ \log 25{,}310 = & \underline{4.40329} \\ \text{Tabular difference} = & 0.00017 \\ & \underline{.4} \\ & 0.000068 \\ \text{Difference to be added} = & 0.00007 \\ \text{Adding this to } 4.40329, \quad \log 25314 = & 4.40336 \end{array}$$

In general, the tabular difference can be found so easily by inspection that it is unnecessary to multiply, as shown in this example. If any multiplication is necessary, it is an easy matter to turn to pages 46 and 47 of the tables, where will be found a table of proportional parts. On page 46, after the number 17 in the column of differences (D), and under 4 (for 0.4), is found 6.8. In the same way we can find any decimal part of a difference.

Exercise 20. Using the Table

Using the table, find the logarithms of the following:

1. 2.	9. 3485.	17. 0.7.	25. 12,340.
2. 20.	10. 4462.	18. 0.75.	26. 12,345.
3. 200.	11. 5581.	19. 0.756.	27. 12,347.
4. 0.002.	12. 7007.	20. 0.7567.	28. 123.47.
5. 2100.	13. 5285.	21. 0.0255.	29. 234.62.
6. 2150.	14. 68.48.	22. 0.0036.	30. 41.327.
7. 2156.	15. 7.926.	23. 0.0009.	31. 56.283.
8. 2.156.	16. 834.8.	24. 0.0178.	32. 0.41282.

33. In a certain computation it is necessary to find the sum of the logarithms of 45.6, 72.8, and 98.4. What is this sum?

34. In a certain computation it is necessary to subtract the logarithm of 3.84 from the sum of the logarithms of 52.8 and 26.5. What is the resulting logarithm?

Perform the following operations:

35. $\log 275 + \log 321 + \log 4.26 + \log 3.87 + \log 46.4$.

36. $\log 2643 + \log 3462 + \log 4926 + \log 5376 + \log 2194$.

37. $\log 51.82 + \log 7.263 + \log 5.826 + \log 218.7 + \log 3275$.

38. $\log 8263 + \log 2179 + \log 3972 - \log 2163 - \log 178$.

39. $\log 37.42 + \log 61.73 + \log 5.823 - \log 1.46 - \log 27.83$.

40. $\log 3.427 + \log 38.46 + \log 723.8 - \log 2.73 - \log 21.68$.

41. In a certain operation it is necessary to find three times log 41.75. What is the resulting logarithm?

42. In a certain operation it is necessary to find one fifth of log 254.8. What is the resulting logarithm?

Perform the following operations:

43. $2 \times \log 3$.	50. $\frac{1}{2} \log 2$.	57. $0.3 \log 431$.
44. $3 \times \log 2$.	51. $\frac{1}{2} \log 2000$.	58. $0.7 \log 43.19$.
45. $3 \times \log 25.6$.	52. $\frac{1}{3} \log 3460$.	59. $0.9 \log 4.007$.
46. $5 \times \log 3.76$.	53. $\frac{1}{3} \log 24.76$.	60. $1.4 \log 5.108$.
47. $4 \times \log 21.42$.	54. $\frac{1}{4} \log 368.7$.	61. $2.3 \log 7.411$.
48. $5 \times \log 346.8$.	55. $\frac{2}{3} \log 41.73$.	62. $\frac{5}{8} \log 16.05$.
49. $12 \times \log 42.86$.	56. $\frac{3}{4} \log 763.8$.	63. $\frac{7}{8} \log 23.43$.

50. Antilogarithm. The number corresponding to a given logarithm is called an *antilogarithm.*

For "antilogarithm of N" it is customary to write "antilog N."

Thus if $\log 25.31 = 1.40329$, antilog $1.40329 = 25.31$. Similarly, we see that antilog $5.40329 = 253{,}100$, and antilog $\bar{2}.40329 = 0.02531$.

51. Finding the Antilogarithm. An antilogarithm is found from the tables by looking for the number corresponding to the given mantissa and placing the decimal point according to the characteristic. For example, consider the following portion of a table:

550 — 600

N	0	1	2	3	4	5	6	7	8	9
550	74 036	74 044	74 052	74 060	74 068	74 076	74 084	74 092	74 099	74 107
551	74 115	74 123	74 131	74 139	74 147	74 15$\underline{5}$	74 162	74 170	74 178	74 186

If the mantissa is given in the table, we find the sequence of the digits of the antilogarithm in the column under **N**. If the mantissa is not given in the table, we interpolate.

1. Find the antilogarithm of 5.74139.

We find 74139 in the table, opposite 551 and under 3. Hence the digits of the number are 5513. Since the characteristic is 5, there are six integral places, and hence the antilogarithm is 551,300. That is,

$$\log 551{,}300 = 5.74139,$$

or

$$\text{antilog } 5.74139 = 551{,}300.$$

2. Find the antilogarithm of $\bar{2}.74166$.

We find 74170 in the table, opposite 551 and under 7.

$$\begin{aligned} \log 0.05517 &= \bar{2}.74170 \\ \log 0.05516 &= \underline{\bar{2}.74162} \\ \text{Tabular difference} &= 0.00008 \end{aligned}$$

Subtracting, we see that, neglecting the decimal point, the tabular difference is 8, and the difference between $\log x$ and $\log 0.05516$ is 4. Hence x is $\frac{4}{8}$ of the way from 0.05516 to 0.05517. Hence $x = 0.055165$.

3. Find the antilogarithm of 7.74053.

We find 74060 in the table, opposite 550 and under 3.

$$\begin{aligned} \log 55{,}030{,}000 &= 7.74060 \\ \log 55{,}020{,}000 &= \underline{7.74052} \\ \text{Tabular difference} &= 0.00008 \end{aligned}$$

Reasoning as before, x is $\frac{1}{8}$ of the way from 55,020,000 to 55,030,000. Hence, to five significant figures, $x = 55{,}021{,}000$.

In general, the interpolation gives only one additional figure correct; that is, with a table like the one above, the sixth figure will not be correct if found by interpolation.

Exercise 21. Antilogarithms

Find the antilogarithms of the following:

1. 0.47712.
2. 3.47712.
3. $\bar{3}.47712$.
4. 2.48359.
5. 4.56844.
6. 1.66276.
7. 2.66978.
8. $\bar{5}.74819$.
9. 3.74076.
10. $\bar{2}.76305$.
11. $\bar{4}.78497$.
12. $\bar{1}.81954$.
13. 0.82575.
14. 0.88081.
15. 9.89237.
16. 7.90282.
17. 0.23305.
18. 1.43144.
19. 2.56838.
20. $\bar{1}.58041$.
21. $\bar{3}.63490$.
22. 4.63492.
23. 0.63994.
24. $\bar{2}.69085$.
25. 8.77425.
26. $\bar{4}.82966$.
27. 3.83547.
28. 2.83604.
29. 4.88960.
30. 2.89523.
31. 3.89858.
32. 0.93223.

33. If the logarithm of the product of two numbers is 2.94210, what is the product of the numbers?

34. If the logarithm of the quotient of two numbers is 0.30103, what is the quotient of the numbers?

35. If we wish to multiply 2857 by 2875, what logarithms do we need? What are these logarithms?

36. If we know that the logarithm of a result which we are seeking is 3.47056, what is that result?

37. If we know that $\log\sqrt{0.000043641}$ is $\bar{3}.81995$, what is the value of $\sqrt{0.000043641}$?

38. If we know that $\log\sqrt[6]{0.076553}$ is $\bar{1}.81400$, what is the value of $\sqrt[6]{0.076553}$?

39. The logarithm of $\sqrt{8322}$ is 1.96012. Find $\sqrt{8322}$ to three decimal places.

40. The logarithm of the cube of 376 is 7.72557. Find the cube of 376 to five significant figures.

41. If we know that $\log 0.003278^2$ is $\bar{5}.03122$, what is the value of 0.003278^2?

42. Find twice log 731, and find the antilogarithm of the result.

43. Find the antilogarithm of the sum of $\log 27.8 + \log 34.6 + \log 367.8$.

Find the antilogarithms of the following:

44. $\log 7 + \log 2 - \log 1.934$.
45. $\log 63 + \log 5.8 - \log 3.415$.
46. $\log 728 + \log 96.8 - \log 2.768$.
47. $5 \log 27.83$.
48. $2.8 \log 5.683$.
49. $\frac{3}{4}(\log 2 + \log 4.2)$.

52. Multiplication by Logarithms. It has been shown (§ 40) that the logarithm of a product is equal to the sum of the logarithms of the numbers. This is of practical value in multiplication.

Find the product of 6.15×27.05.

From the tables,
$$\begin{aligned}\log 6.15 &= 0.78888\\ \log 27.05 &= \underline{1.43217}\\ \log x &= 2.22105\end{aligned}$$

Interpolating to find the value of x, we have
$$\begin{aligned}\log 166.4 &= 2.22115\\ \log 166.3 &= \underline{2.22089}\\ &26\end{aligned} \qquad \begin{aligned}\log x &= 2.22105\\ \log 166.3 &= \underline{2.22089}\\ &16\end{aligned}$$

Annexing to 166.3 the fraction $\frac{16}{26}$, we have
$$\begin{aligned}x &= 166.3\tfrac{16}{26}\\ &= 166.36,\end{aligned}$$
the interpolation not being exact beyond one figure.

If we perform the actual multiplication, we have $6.15 \times 27.05 = 166.3575$, or 166.36 to two decimal places.

Exercise 22. Multiplication by Logarithms

Using logarithms, find the following products:

1. 2×5.
2. 4×6.
3. 3×5.
4. 5×7.
5. 2×4.
6. 3×7.
7. 2×6.
8. 3×6.
9. 7×8.
10. 2×9.
11. 2×50.
12. 40×60.
13. 3×500.
14. 50×70.
15. 2×4000.
16. 30×700.
17. 200×60.
18. 30×600.
19. $7 \times 80{,}000$.
20. 200×900.
21. 35.8×28.9.
22. 52.7×41.6.
23. 2.75×4.84.
24. 5.25×3.86.
25. 14.26×42.35.
26. 43.28×29.64.
27. 529.6×348.7.
28. 240.8×46.09.
29. 34.81×46.25.
30. 5028×3.472.

31. Taking the circumference of a circle to be 3.14 times the diameter, find the circumference of a steel shaft of diameter 5.8 in.

32. Taking the ratio of the circumference to the diameter as given in Ex. 31, find the circumference of a water tank of diameter 36 ft.

Using logarithms, find the following products:

33. $2 \times 3 \times 5 \times 7$.
34. $3 \times 5 \times 7 \times 9$.
35. $5 \times 7 \times 11 \times 13$.
36. $43.8 \times 26.9 \times 32.8$.
37. $527.6 \times 283.4 \times 4.196$.
38. $7.283 \times 6.987 \times 5.437$.

53. Negative Characteristic. Since the mantissa is always positive (§ 45), care has to be taken in adding or subtracting logarithms in which a negative characteristic may occur. In all such cases it is better to separate the characteristics from the mantissas, as shown in the following illustrations:

1. Add the logarithms $\bar{2}.81764$ and 1.41283.

Separating the negative characteristic from its mantissa, we have

$$\begin{aligned} \bar{2}.81764 &= 0.81764 - 2 \\ 1.41283 &= \underline{1.41283} \\ \text{Adding, we have}\quad & \ 2.23047 - 2 \\ &= 0.23047 \end{aligned}$$

2. Add the logarithms $\bar{4}.21255$ and $\bar{2}.96245$.

Separating both negative characteristics from the mantissas, we have

$$\begin{aligned} \bar{4}.21255 &= 0.21255 - 4 \\ \bar{2}.96245 &= \underline{0.96245 - 2} \\ \text{Adding, we have}\quad & \ 1.17500 - 6 \\ &= \bar{5}.17500 \end{aligned}$$

Exercise 23. Negative Characteristics

Add the following logarithms:

1. $2.41283 + 5.27681$.
2. $\bar{2}.41283 + 5.27681$.
3. $\bar{2}.41283 + \bar{5}.27681$.
4. $0.38264 + \bar{4}.71233$.
5. $0.57121 + \bar{1}.42879$.
6. $\bar{2}.63841 + 1.36158$.
7. $\bar{2}.41238 + \bar{3}.62701$.
8. $\bar{5}.58623 + 6.41387$.
9. $\bar{6}.41382 + 7.58617$.
10. $\bar{4}.22334 + 3.77666$.

Using logarithms, find the following products:

11. 256×4875.
12. 2.56×48.75.
13. 0.256×0.4875.
14. 0.0256×0.004875.
15. 0.1275×0.03428.
16. 0.2763×0.4134.
17. 0.00025×0.00125.
18. 0.725×0.3465.
19. 0.256×0.0875.
20. 0.037×0.00425.
21. 47.26×0.02755.
22. 296.8×0.1283.
23. $45{,}650 \times 0.0725$.
24. $127{,}400 \times 0.00355$.
25. Given $\sin 25.75° = 0.4344$, find $52.8 \sin 25.75°$.
26. Given $\cos 37.25° = 0.7960$, find $42.85 \cos 37.25°$.
27. Given $\tan 30° 50' 30'' = 0.5971$, find $27.65 \tan 30° 50' 30''$.

54. Division by Logarithms. It has been shown (§ 41) that the logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

Care must be taken that the mantissa in subtraction does not become negative (§ 45).

1. Using logarithms, divide 17.28 by 1.44.

From the tables,

$$\begin{aligned} \log 17.28 &= 1.23754 \\ \log 1.44 &= \underline{0.15836} \\ & 1.07918 \\ &= \log 12 \end{aligned}$$

Hence $17.28 \div 1.44 = 12$.

2. Using logarithms, divide 2603.5 by 0.015998.

$$\begin{aligned} \log 2603.5 &= 3.41556 \\ \log 0.015998 &= \underline{\bar{2}.20407} \end{aligned}$$

Arranging these in a form more convenient for subtracting, we have

$$\begin{aligned} \log 2603.5 &= 3.41556 \\ \log 0.015998 &= \underline{0.20407 - 2} \\ & 3.21149 + 2 \\ &= 5.21149 = \log 162{,}740 \end{aligned}$$

Hence $2603.5 \div 0.015998 = 162{,}740$.

3. Using logarithms, divide 0.016502 by 127.41.

$$\begin{aligned} \log 0.016502 &= \bar{2}.21753 = 8.21753 - 10 \\ \log 127.41 &= 2.10520 = \underline{2.10520 } \\ & 6.11233 - 10 \\ & = \bar{4}.11233 = \log 0.00012952 \end{aligned}$$

Hence $0.016502 \div 127.41 = 0.00012952$.

Here we increased $\bar{2}.21753$ by 10 and decreased the sum by 10. We might take any other number that would make the highest order of the minuend larger than the corresponding order of the subtrahend, but it is a convenient custom to take 10 or the smallest multiple of 10 that will serve the purpose.

4. Using logarithms, divide 0.000148 by 0.022922.

$$\begin{aligned} \log 0.000148 &= \bar{4}.17026 = 16.17026 - 20 \\ \log 0.022922 &= \bar{2}.36025 = \underline{8.36025 - 10} \\ & \phantom{= \bar{2}.36025 =} 7.81001 - 10 \\ & \phantom{= \bar{2}.36025} = \bar{3}.81001 = \log 0.0064567 \end{aligned}$$

Hence $0.000148 \div 0.022922 = 0.0064567$.

5. Using logarithms, divide 0.2548 by 0.05513.

$$\begin{aligned} \log 0.2548 &= \bar{1}.40620 = 9.40620 - 10 \\ \log 0.05513 &= \bar{2}.74139 = \underline{8.74139 - 10} \\ & \phantom{= \bar{2}.74139 =} 0.66481 \\ & \phantom{= \bar{2}.74139} = \log 4.6218 \end{aligned}$$

Hence $0.2548 \div 0.05513 = 4.6218$.

Exercise 24. Division by Logarithms

Add the following logarithms:

1. $\overline{2}.14755 + 3.82764$.
2. $\overline{4}.07256 + 1.58822$.
3. $0.21783 + \overline{1}.46835$.
4. $0.41722 + \overline{3}.28682$.
5. $\overline{4}.18755 + \overline{2}.81245$.
6. $\overline{6}.28742 + \overline{3}.41258$.
7. $\overline{4}.21722 + \overline{4}.78278$.
8. $\overline{5}.28720 + \overline{3}.71280$.

9. Find the sum of $\overline{2}.41280$, $\overline{4}.17623$, $\overline{5}.26453$, 0.21020, 7.36423, 2.63577, $\overline{6}.41323$, and 3.28740.

From the first of these logarithms subtract the second:

10. 0.21250, $\overline{2}.21250$.
11. 0.17286, $\overline{3}.27286$.
12. 2.34222, $\overline{5}.44222$.
13. 3.14725, $\overline{1}.25625$.
14. $\overline{4}.17325$, $\overline{2}.17325$.
15. $\overline{5}.82340$, $\overline{3}.71120$.
16. $\overline{3}.14286$, $\overline{1}.14000$.
17. $\overline{3}.27283$, $\overline{5}.56111$.

Using logarithms, divide as follows:

18. $10 \div 2$.
19. $15 \div 3$.
20. $15 \div 5$.
21. $12 \div 3$.
22. $12 \div 4$.
23. $60 \div 12$.
24. $75 \div 25$.
25. $125 \div 25$.
26. $25,284 \div 301$.
27. $51,742 \div 631$.
28. $47,348 \div 623$.
29. $19,224 \div 540$.
30. $37,960 \div 520$.
31. $84,640 \div 920$.
32. $65,100 \div 620$.
33. $45,990 \div 730$.
34. $59.29 \div 0.77$.
35. $2.451 \div 190$.
36. $851.4 \div 0.66$.
37. $0.98902 \div 99$.
38. $0.41831 \div 5.9$.
39. $0.08772 \div 4.3$.
40. $0.02275 \div 0.35$.
41. $0.02736 \div 0.057$

Using logarithms, divide to four significant figures:

42. $15 \div 7$.
43. $7 \div 15$.
44. $0.7 \div 150$.
45. $26.4 \div 13.8$.
46. $4.21 \div 3.75$.
47. $63.25 \div 4.92$.
48. $17.625 \div 3.4$.
49. $43.826 \div 0.72$.
50. $5.483 \div 8.4$.

Taking log 3.1416 as 0.49715 and interpolating for six figures on the same principle as for five, find the diameters of circles with circumferences as follows:

51. 62.832.
52. 157.08.
53. 2199.12.
54. 2513.28.
55. 28,274.2.
56. 34,557.6.
57. 376,992.
58. 0.031416.

59. By using logarithms find the product of 41.74×20.87, and the quotient of $41.74 \div 20.87$.

55. Cologarithm. The logarithm of the reciprocal of a number is called the *cologarithm* of the number.

For "cologarithm of N" it is customary to write "colog N."

By definition $\operatorname{colog} x = \log\frac{1}{x} = \log 1 - \log x$ (§ 41). But $\log 1 = 0$.

Hence we have $\qquad \operatorname{colog} x = -\log x.$

To avoid a negative mantissa (§ 45) it is customary to consider that

$$\operatorname{colog} x = 10 - \log x - 10,$$

since $10 - \log x - 10$ is the same as $-\log x$.

For example, $\operatorname{colog} 2 = -\log 2 = 10 - \log 2 - 10$
$= 10 - 0.30103 - 10$
$= 9.69897 - 10 = \bar{1}.69897.$

56. Use of the Cologarithm. Since to divide by a number is the same as to multiply by its reciprocal, *instead of subtracting the logarithm of a divisor we may add its cologarithm.*

The cologarithm of a number is easily written by looking at the logarithm in the table. Thus, since $\log 20 = 1.30103$, we find colog 20 by subtracting this from $10.00000 - 10$. To do this we begin at the left and subtract the number represented by each figure from 9, except the right-hand significant figure, which we subtract from 10. In full form we have

$$\begin{array}{rlcccccccl} & 10.00000 - 10 = & 9. & 9 & 9 & 9 & 9 & 10 & -10 & \\ \log 20 = & 1.30103 = & 1. & 3 & 0 & 1 & 0 & 3 & & \\ \hline \operatorname{colog} 20 = & & 8. & 6 & 9 & 8 & 9 & 7 & -10 & = \bar{2}.69897 \end{array}$$

Similarly, we may find colog 0.03952 thus:

$$\begin{array}{rlcccccccl} & 10.00000 - 10 = & 9. & 9 & 9 & 9 & 9 & 10 & -10 & \\ \log 0.03952 = & \bar{2}.59682 = & 8. & 5 & 9 & 6 & 8 & 2 & -10 & \\ \hline \operatorname{colog} 0.03952 = & & 1. & 4 & 0 & 3 & 1 & 8 & & = 1.40318 \end{array}$$

Practically, of course, we would find log 0.03952 and subtract mentally.

Exercise 25. Cologarithms

Write the cologarithms of the following numbers:

1. 25.	5. 3751.	9. 0.5.	13. 3.007.
2. 130.	6. 427.3.	10. 0.72.	14. 62.09.
3. 27.4.	7. 51.61.	11. 0.083.	15. 0.0006.
4. 5.83.	8. 7.213.	12. 0.00726.	16. 0.00007.

17. What number has 0 for its cologarithm?

18. What number has 1 for its cologarithm?

19. What number has ∞ for its cologarithm?

20. Find the number whose cologarithm equals its logarithm.

57. Advantages of the Cologarithm. If, as is not infrequently the case in the computations of trigonometry and physics, we have the product of two or more numbers to be divided by the product of two or more different numbers, the cologarithm is of great advantage.

Using logarithms and cologarithms, simplify the expression

$$\frac{17.28 \times 6.25 \times 16.9}{1.44 \times 0.25 \times 1.3}.$$

This is so chosen that we can easily verify the answer by cancellation.
By logarithms we have,

$$\begin{aligned}
\log 17.28 &= 1.23754\\
\log 6.25 &= 0.79588\\
\log 16.9 &= 1.22789\\
\operatorname{colog} 1.44 &= 9.84164 - 10\\
\operatorname{colog} 0.25 &= 0.60206\\
\operatorname{colog} 1.3 &= \underline{9.88606 - 10}\\
&\ 3.59107 \qquad = \log 3900.1
\end{aligned}$$

In a long computation the fifth figure may be in error.

Exercise 26. Use of Cologarithms

Using cologarithms, find the value of the following to five figures:

1. $\dfrac{3 \times 2}{4 \times 1.5}$.
2. $\dfrac{8 \times 9}{3 \times 4}$.
3. $\dfrac{6 \times 12}{3 \times 8}$.
4. $\dfrac{4 \times 24}{12 \times 16}$.
5. $\dfrac{12 \times 15}{9 \times 20}$.
6. $\dfrac{12 \times 28}{8 \times 21}$.
7. $\dfrac{3 \times 22}{18 \times 33}$.
8. $\dfrac{11 \times 13}{17 \times 19}$.
9. $\dfrac{15 \times 17}{11 \times 13}$.
10. $\dfrac{172.8 \times 1.44}{0.288 \times 0.864}$.
11. $\dfrac{57.5 \times 0.64}{1.25 \times 320}$.
12. $\dfrac{1.28 \times 13.41}{1.49 \times 6.4}$.
13. $\dfrac{5.48 \times 0.198}{3.96 \times 27.4}$.
14. $\dfrac{1.176 \times 10.22}{14.6 \times 3.92}$.
15. $\dfrac{3 \times 11 \times 17}{7 \times 13}$.
16. $\dfrac{16 \times 23}{3 \times 7 \times 41}$.
17. $\dfrac{23 \times 39 \times 47}{17 \times 33 \times 53}$.
18. $\dfrac{0.2 \times 0.3}{0.11 \times 17\frac{1}{2}}$.
19. $\dfrac{435 \times 0.2751}{2.83 \times 1.045}$.
20. $\dfrac{50.05 \times 2.742}{381.4 \times 2.461}$.
21. $\dfrac{50730 \times 2.875}{34.48 \times 1.462}$.
22. $\dfrac{3.427 \times 0.7832}{3.1416 \times 0.0081}$.
23. $\dfrac{27.98 \times 32.05}{0.48 \times 0.00062}$.
24. $\dfrac{2.1 \times 0.3 \times 0.11}{17 \times 0.05}$.
25. $\dfrac{1.1 \times 3.003}{0.2 \times 0.07112}$.
26. $\dfrac{0.0347 \times 0.117}{3 \times 11 \times 170}$.
27. $\dfrac{528.4 \times 1.001}{7.03 \times 0.7281}$.

58. Raising to a Power. It has been shown (§ 42) that the logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent.

1. Find by logarithms the value of 11^3.

From the tables, $\log 11 = 1.04139$
Multiplying by 3, $\underline{3}$
$\log 11^3 = 3.12417$
$= \log 1331.0$

That is, $11^3 = 1331.0$, to five figures. Of course we see that $11^3 = 1331$ exactly, log 1331 being 3.12418. The last figure in $\log 11^3$ as found in the above multiplication is therefore not exact, as is frequently the case in such computations.

As usual, care must be taken when a negative characteristic appears.

2. Find by logarithms the value of 0.2413^5.

From the tables, $\log 0.2413 = 0.38256 - 1$
Multiplying by 5, $\underline{5}$
$\log 0.2413^5 = 1.91280 - 5$
$= \bar{4}.91280$
$= \log 0.00081808$

Hence $0.2413^5 = 0.00081808$, to five significant figures.

As on page 18, we use the expression "significant figures" to indicate the figures after the zeros at the left, even though some of these figures are zero.

Exercise 27. Raising to Powers

By logarithms, find the value of each of the following to five significant figures:

1. 2^2.	9. 1^{10}.	17. 25^3.	25. 1.1^8.	33. 12.55^2.
2. 2^3.	10. 7^9.	18. 25^7.	26. 2.1^7.	34. 34.75^3.
3. 2^5.	11. 9^7.	19. 125^2.	27. 0.1^{12}.	35. 1.275^3.
4. 2^{10}.	12. 8^8.	20. 625^3.	28. 0.2^{11}.	36. 0.1254^3.
5. 3^4.	13. 11^7.	21. 1750^5.	29. 0.7^8.	37. 0.4725^5.
6. 3^6.	14. 15^6.	22. 2775^2.	30. 0.07^6.	38. 0.01234^2.
7. 4^3.	15. 1.5^6.	23. 3146^3.	31. 0.37^4.	39. 0.00275^2.
8. 5^3.	16. 17^4.	24. 4135^4.	32. 5.37^3.	40. 0.000355^2.

41. If $\log \pi = 0.49715$, what is the value of π^2? of π^3?

42. Using $\log \pi$ as in Ex. 41, what is the value of πr when $r = 7$? of πr^2 when $r = 7$? of $\frac{4}{3}\pi r^3$ when $r = 9$?

59. Fractional Exponent. It has been shown (§ 43) that the logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root. This law may, however, be combined with that of § 58, since $a^{\frac{1}{2}}$ means $\sqrt{a}$, and $a^{\frac{2}{3}}$ means $\sqrt[3]{a^2}$. The law of § 58 therefore applies to roots or to powers of roots, the exponent simply being considered fractional.

1. Find by logarithms the value of $\sqrt{4}$, or $4^{\frac{1}{2}}$.

From the tables, $\log 4 = 0.60206$

Dividing by 2, $2\overline{)0.60206}$

$$\log \sqrt{4}, \text{ or } \log 4^{\frac{1}{2}}, = 0.30103$$
$$= \log 2$$

Hence $\sqrt{4}$, or $4^{\frac{1}{2}}$, is 2.

2. Find by logarithms the value of $8^{\frac{2}{3}}$.

From the tables, $\log 8 \;= 0.90309$

Multiplying by $\frac{2}{3}$, $\log 8^{\frac{2}{3}} = 0.60206$

$$= \log 4$$

Therefore $8^{\frac{2}{3}} = 4$.

3. Find by logarithms the value of $0.127^{\frac{1}{5}}$.

From the tables, $\log 0.127 = 0.10380 - 1$.

Since we cannot divide -1 by 5 and get an integral quotient for the new characteristic, we add 4 and subtract 4 and then have

$$\log 0.127 = 4.10380 - 5$$

Dividing by 5, $\log 0.127^{\frac{1}{5}} = 0.82076 - 1$

$$= \log 0.66185$$

Hence $0.127^{\frac{1}{5}}$, or $\sqrt[5]{0.127}$, is 0.66185.

We might have written $\log 0.127 = 9.10380 - 10$, $14.10380 - 15$, and so on.

Exercise 28. Extracting Roots

By logarithms, find the value of each of the following:

1. $\sqrt{2}$.	5. $2^{\frac{1}{5}}$.	9. $\sqrt{11}$.	13. $0.3^{\frac{1}{2}}$.	17. $127.8^{\frac{5}{8}}$.
2. $\sqrt[3]{5}$.	6. $3^{\frac{3}{4}}$.	10. $\sqrt[3]{3}$.	14. $0.05^{\frac{1}{3}}$.	18. $2.475^{\frac{3}{4}}$.
3. $\sqrt[7]{7}$.	7. $8^{\frac{5}{6}}$.	11. $\sqrt[3]{22}$.	15. $0.0175^{\frac{2}{3}}$.	19. $5.135^{\frac{5}{6}}$.
4. $\sqrt[15]{25}$.	8. $7^{\frac{4}{7}}$.	12. $\sqrt[25]{100}$.	16. $0.0325^{\frac{4}{5}}$.	20. $0.00125^{\frac{7}{8}}$.

21. If $\log \pi = 0.49715$, what is the value of $\sqrt{\pi}$? of $\sqrt[3]{\pi}$?

22. Using the value of $\log \pi$ given in Ex. 21, what is the value of $\pi^{\frac{1}{4}}$? of $\pi^{\frac{2}{3}}$? of $\pi^{\frac{3}{2}}$? of $\pi^{-\frac{3}{4}}$? of $\pi^{-\frac{4}{5}}$? of $\pi^{-0.2}$?

60. Exponential Equation. An equation in which the unknown quantity appears in an exponent is called an *exponential equation.*

Exponential equations may often be solved by the aid of logarithms.

1. Given $5^x = 625$, find by logarithms the value of x.

Taking the logarithms of both sides, we have (§ 42)

$$x \log 5 = \log 625$$

Whence

$$x = \frac{\log 625}{\log 5}$$

$$= \frac{2.79588}{0.69897} = 4$$

Check. $5^4 = 625.$

In all such cases bear in mind that one logarithm must actually be divided by the other. If we wished to perform this division by means of logarithms, we should have to take the logarithm of 2.79588 and the logarithm of 0.69897, subtract the second logarithm from the first, and then find the antilogarithm.

We may apply this principle to certain simultaneous equations.

2. Solve this pair of simultaneous equations

$$2^x \cdot 3^y = 72 \tag{1}$$

$$4^x \cdot 27^y = 46{,}656 \tag{2}$$

Taking the logarithms of both sides, we have (§§ 40, 42)

$$x \log 2 + y \log 3 = \log 72, \tag{3}$$

and

$$x \log 4 + y \log 27 = \log 46{,}656. \tag{4}$$

Then, since

$$\log 4 = \log 2^2 = 2 \log 2,$$

and

$$\log 27 = \log 3^3 = 3 \log 3,$$

we have

$$2x \log 2 + 3y \log 3 = \log 46{,}656. \tag{5}$$

Eliminating x by multiplying equation (3) by 2 and subtracting from equation (5), we have

$$y = \frac{\log 46656 - 2 \log 72}{\log 3}$$

$$= \frac{4.66890 - 2 \times 1.85733}{0.47712}$$

$$= \frac{0.95424}{0.47712} = 2$$

We may substitute this value of y in (1), divide by 3^2, and then find x by taking the logarithms of both sides. It will be found that $x = 3$.

We may check by substituting in (2).

In the same way, equations involving three or more unknown quantities may be solved. Although the exponential equation is valuable in algebra, as in the solution of Exs. 22, 23, 25, and 26 of Exercise 29, we rarely have need of it in trigonometry.

Exercise 29. Exponential Equations

By logarithms, solve the following exponential equations:

1. $2^x = 8$.
2. $3^x = 81$.
3. $5^x = 625$.
4. $4^x = 256$.
5. $11^x = 1331$.
6. $2^x = 19$.
7. $3^x = 75$.
8. $5^x = 1000$.
9. $4^x = 2560$.
10. $11^x = 1500$.
11. $2^{-x} = \frac{1}{8}$.
12. $2^{-x} = 0.1$.
13. $0.3^{-x} = 0.9$.
14. $2^{x+1} = 3^{x-1}$.
15. $9^{x+5} = 53{,}143$.

Solve the following simultaneous equations:

16. $a^{x+y} = a^4$, $a^{x-y} = a^2$
17. $m^{2x+y} = m^{11}$, $n^{3x-y} = n^{14}$
18. $3^x \cdot 4^y = 12$, $5^x \cdot 7^y = 35$
19. $2^x \cdot 3^y = 36$, $4^x \cdot 5^y = 400$
20. $2^x \cdot 5^y = 200$, $3^x \cdot 3^y = 243$
21. $2^x \cdot 8^y = 256$, $8^x \cdot 32^y = 65{,}536$

Solve the following equations by logarithms:

22. $a = p(1+r)^x$.
23. $l = ar^{x-1}$.
24. $2^{x^2+2x} = 8$.
25. $a = p(1+rt)^x$.
26. $s(r-1) = ar^x - a$.
27. $3^{x^2-x+1} = 27$.

Perform the following operations by logarithms:

28. $\dfrac{2.47 \times 84.96}{34.8 \times 96.55}$.
29. $\sqrt[4]{\dfrac{42.4 \times 0.075}{3.64 \times 0.009}}$.
30. $\left(\dfrac{5.75 \times 3.428}{59.62 \times 48.08}\right)^{\frac{2}{3}}$.
31. $\sqrt[5]{\left(\dfrac{0.07 \times 0.00964}{3.426 \times 0.875}\right)^2}$.

32. To what power must 7 be raised to equal 117,649?

33. To what power must a be raised to equal b?

34. To what power must 5 be raised to equal n?

35. Find the value of x when $\sqrt[x]{9} = 3$; when $\sqrt[x]{2} = 1.1$; when $\sqrt[x]{2} = 1.414$; when $\sqrt[x]{3} = 1.73$.

36. Find the value of x when $\sqrt[x]{3} = 3$; when $\sqrt[x]{a} = b$; when $\sqrt[x]{a} = a$; when $\sqrt[x]{1331} = 11$; when $\sqrt[x]{20736} = 12$.

37. Solve the equations

$$\sqrt[x]{y} = a$$
$$\sqrt[x+1]{y} = b$$

38. What value of x satisfies the equation $a^{\frac{1}{x^2+2x+4}} = \sqrt[3]{a}$?

61. Logarithms of the Functions. Since computations involving trigonometric functions are often laborious, they are generally performed by the aid of logarithms. For this reason tables have been prepared giving the logarithms of the sine, cosine, tangent, and cotangent of the various angles from 0° to 90° at intervals of 1′. The functions of angles greater than 90° are defined and discussed later in this work when the need for them arises.

Logarithms of the secant and cosecant are usually not given for the reason that the secant is the reciprocal of the cosine, and the cosecant is the reciprocal of the sine. Instead of multiplying by $\sec x$, for example, we may divide by $\cos x$; and when we are using logarithms one operation is as simple as the other, since multiplication requires the addition of a logarithm and division requires the addition of a cologarithm.

In order to avoid negative characteristics the characteristic of every logarithm of a trigonometric function is printed 10 too large, and hence 10 must be subtracted from it.

Practically this gives rise to no confusion, for we can always tell by a result if a logarithm is 10 too large, since it would give an antilogarithm with 10 integral places more than it should have. For example, if we are measuring the length of a lake in miles, and find 10.30103 as the logarithm of the result, we see that the characteristic must be much too large, since this would make the lake 20,000,000,000 mi. long.

It would be possible to print $\bar{2}.97496$ for $\log \sin 5° 25'$, instead of 8.97496, which is 10 too large. It would be more troublesome, however, for the eye to detect the negative sign than it would be to think of the characteristic as 10 too large.

On pages 56–77 of the tables the characteristic remains the same throughout each column, and is therefore printed only at the top and bottom, except in the case of pages 58 and 77. Here the characteristic changes one unit at the places marked with the bars. By a little practice, such as is afforded on pages 61 and 62 of the text, the use of the tables will become clear.

On account of the rapid change of the sine and tangent for very small angles $\log \sin x$ is given for every second from 0″ to 3′ on page 49 of the tables, and $\log \tan x$ has identically the same values to five decimal places. The same table, read upwards, gives the $\log \cos x$ for every second from 89° 57′ to 90°. Also $\log \sin x$, $\log \tan x$, and $\log \cos x$ are given, on pages 50–55 of the tables, for every 10″ from 0″ to 2°. Reading from the foot of the page, the cofunctions of the complementary angles are given.

On pages 56–77 of the tables, $\log \sin x$, $\log \cos x$, $\log \tan x$, and $\log \cot x$ are given for every minute from 1° to 89°. Interpolation in the usual manner (page 31) gives the logarithmic functions for every second from 1° to 89°.

62. Use of the Tables. The tables are used in much the same way as the tables of natural functions.

For example,	$\log\sin\ 5^\circ\ 25'$	$= 8.97496 - 10$	Page 58
	$\log\tan 40^\circ\ 55'$	$= 9.93789 - 10$	Page 75
	$\log\cos 52^\circ\ 20'$	$= 9.78609 - 10$	Page 74
	$\log\cot 88^\circ\ 59'$	$= 8.24910 - 10$	Page 56
	$\log\sin\ 0^\circ\ 28'\ 40''$	$= 7.92110 - 10$	Page 51
	$\log\sin\ 0^\circ\ 1'\ 52''$	$= 6.73479 - 10$	Page 49

Furthermore, if $\log\cot x = 9.55910 - 10$, then $x = 70^\circ\ 5'$. Page 65

Interpolation is performed in the usual manner, whether the angles are expressed in the sexagesimal system or decimally.

1. Find $\log\sin 19^\circ\ 50'\ 30''$.

From the tables, $\log\sin 19^\circ\ 50' = 9.53056 - 10$, and the tabular difference is 36. We must therefore add $\frac{30}{60}$ of 36 to the mantissa, in the proper place. We therefore add 0.00018, and have $\log\sin 19^\circ\ 50'\ 30'' = 9.53074 - 10$.

2. Find $\log\tan 39.75^\circ$.

From the tables, $\log\tan 39.7^\circ = 9.91919 - 10$, and the tabular difference is 154. We therefore add 0.5 of 154 to the mantissa, in the proper place. Adding 0.00077, we have $\log\tan 39.75^\circ = 9.91996 - 10$.

Special directions in the case of very small angles are given on page 49 of the tables. It should be understood, however, that we rarely use angles involving seconds except in astronomy.

If the function is decreasing, care must be taken to subtract instead of add in making an interpolation.

3. Find $\log\cos 43^\circ\ 45'\ 15''$.

From the tables, $\log\cos 43^\circ\ 45' = 9.85876 - 10$, and the tabular difference is 12. Taking $\frac{15}{60}$ of 12, or $\frac{1}{4}$ of 12, we have 0.00003 to be *subtracted*.

Therefore $\log\cos 43^\circ\ 45'\ 15'' = 9.85873 - 10$.

4. Given $\log\cot x = 0.19268$, find x.

From the tables, $\log\cot 32^\circ\ 41' = 10.19275 - 10 = 0.19275$.

The tabular difference is 28, and the difference between the logarithm 0.19275 and the given logarithm is 7, in each case hundred-thousandths. Hence there is an angular difference of $\frac{7}{28}$ of $1'$, or $\frac{1}{4}$ of $1'$, or $15''$. Since the angle increases as the cotangent decreases, and 0.19268 is less than $10.19275 - 10$, we have to add $15''$ to $32^\circ\ 41'$, whence $x = 32^\circ\ 41'\ 15''$.

5. Given $\log\tan x = 0.26629$, find x.

From the tables, $\log\tan 61^\circ\ 33' = 10.26614 - 10 = 0.26614$.

The tabular difference is 30, and the difference between the logarithm 0.26614 and the given logarithm is 15, in each case hundred-thousandths. Hence there is an angular difference of $\frac{15}{30}$ of $1'$, or $30''$. Since $f(x)$ is increasing in this case, and x is also increasing, we add $30''$ to $61^\circ\ 33'$. Hence $x = 61^\circ\ 33'\ 30''$.

Exercise 30. Use of the Tables

Find the value of each of the following:

1. $\log\sin 27°$.
2. $\log\sin 69°$.
3. $\log\cos 36°$.
4. $\log\cos 48°$.
5. $\log\tan 75°$.
6. $\log\tan 12°$.
7. $\log\cot 15°$.
8. $\log\cot 78°$.
9. $\log\sin 9° 15'$.
10. $\log\cos 8° 27'$.
11. $\log\tan 7° 56'$.
12. $\log\cot 82° 4'$.
13. $\log\sin 4.5°$.
14. $\log\cos 7.25°$.
15. $\log\tan 9.75°$.
16. $\log\cos 42° 45''$.
17. $\log\tan 26° 15''$.
18. $\log\cot 38° 30''$.
19. $\log\sin 21° 10' 4''$.
20. $\log\sin 68° 49' 56''$.
21. $\log\cos 15° 17' 3''$.
22. $\log\cos 74° 42' 57''$.
23. $\log\tan 17° 2' 10''$.
24. $\log\tan 26° 3' 4''$.
25. $\log\cot 48° 4' 5''$.
26. $\log\cot 4° 10' 7''$.
27. $\log\sin 34° 30''$.
28. $\log\sin 27.45°$.
29. $\log\tan 56.35°$.
30. $\log\cos 48.26°$.
31. $\log\sin 0° 1' 7''$.
32. $\log\sin 1° 2' 5''$.
33. $\log\tan 0° 2' 8''$.
34. $\log\tan 2° 7' 7''$.
35. $\log\cos 89° 50' 10''$
36. $\log\cos 89° 10' 45''$.
37. $\log\cot 89° 15' 12''$.
38. $\log\cot 89° 25' 15''$.
39. $\log\sin 1° 1' 1''$.
40. $\log\cos 88° 58' 59''$.
41. $\log\tan 2° 27' 25''$.
42. $\log\cot 87° 32' 45''$.
43. $\log\sin 12° 12' 12''$.
44. $\log\cos 77° 47' 48''$.
45. $\log\tan 68° 6' 43''$.

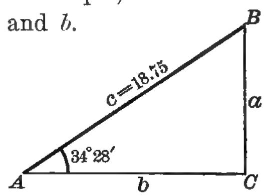
Find the value of x, given the following logarithms, each of which is 10 too large:

46. $\log\sin x = 9.11570$.
47. $\log\sin x = 9.72843$.
48. $\log\sin x = 9.93053$.
49. $\log\sin x = 9.99866$.
50. $\log\cos x = 9.99866$.
51. $\log\cos x = 9.93053$.
52. $\log\cos x = 9.71705$.
53. $\log\cos x = 9.80320$.
54. $\log\tan x = 9.90889$.
55. $\log\tan x = 10.30587$.
56. $\log\tan x = 10.64011$.
57. $\log\cot x = 9.28865$.
58. $\log\cot x = 9.56107$.
59. $\log\sin x = 9.53871$.
60. $\log\sin x = 9.72868$.
61. $\log\sin x = 9.88150$.
62. $\log\sin x = 9.89530$.
63. $\log\cos x = 9.90151$.
64. $\log\cos x = 9.80070$.
65. $\log\cos x = 9.99483$.
66. $\log\tan x = 9.18854$.
67. $\log\tan x = 10.18750$.
68. $\log\tan x = 10.06725$.
69. $\log\cot x = 10.10134$.
70. $\log\cot x = 11.44442$.
71. $\log\cot x = 7.49849$.

CHAPTER IV

THE RIGHT TRIANGLE

63. Given an Acute Angle and the Hypotenuse. In § 30 the solution of the right triangle was considered when an acute angle and the hypotenuse are given. We may now consider this case and the following cases with the aid of logarithms. For example, given $A = 34° 28'$, $c = 18.75$, find B, a, and b.



1. $B = 90° - A = 55° 32'$.

2. $\frac{a}{c} = \sin A$; $\therefore a = c \sin A$.

3. $\frac{b}{c} = \cos A$; $\therefore b = c \cos A$.

$$\begin{aligned} \log a &= \log c + \log \sin A \\ \log c &= 1.27300 \\ \log \sin A &= \underline{9.75276 - 10} \\ \log a &= 1.02576 \\ \therefore\ a &= 10.611 \\ &= 10.61 \end{aligned}$$

$$\begin{aligned} \log b &= \log c + \log \cos A \\ \log c &= 1.27300 \\ \log \cos A &= \underline{9.91617 - 10} \\ \log b &= 1.18917 \\ \therefore\ b &= 15.459 \\ &= 15.46 \end{aligned}$$

Check. $10.61^2 + 15.46^2 = 351.58$, and $18.75^2 = 351.56$.

This solution may be compared with the one on page 35. In this case there is a gain in using logarithms, since we avoid two multiplications by 18.75.

The result is given to four figures (two decimal places) only, the length of c having been given to four figures (two decimal places) only, and this probably being all that is desired. In general, *the result cannot be more nearly accurate than data derived from measurement.*

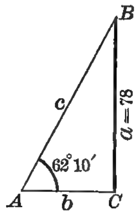
Consider also the case in which $A = 72° 27' 42''$, $c = 147.35$, to find B, a, and b as above.

$$\begin{aligned} \log a &= \log c + \log \sin A \\ \log c &= 2.16835 \\ \log \sin A &= \underline{9.97933 - 10} \\ \log a &= 2.14768 \\ \therefore\ a &= 140.50 \end{aligned}$$

$$\begin{aligned} \log b &= \log c + \log \cos A \\ \log c &= 2.16835 \\ \log \cos A &= \underline{9.47906 - 10} \\ \log b &= 1.64741 \\ \therefore\ b &= 44.403 \end{aligned}$$

Check. What convenient check can be applied in this case?

64. Given an Acute Angle and the Opposite Side. For example, given $A = 62° 10'$, $a = 78$, find B, b, and c.



1. $B = 90° - A = 27° 50'$.

2. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.

3. $\frac{a}{c} = \sin A$;

$\therefore a = c \sin A$, and $c = \frac{a}{\sin A}$.

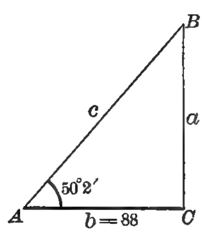
$$\begin{aligned} \log b &= \log a + \log \cot A \\ \log a &= 1.89209 \\ \log \cot A &= \underline{9.72262 - 10} \\ \log b &= 1.61471 \\ \therefore b &= 41.182 \\ &= 41.18 \end{aligned}$$

$$\begin{aligned} \log c &= \log a + \operatorname{colog} \sin A \\ \log a &= 1.89209 \\ \operatorname{colog} \sin A &= \underline{0.05340} \\ \log c &= 1.94549 \\ \therefore c &= 88.204 \\ &= 88.20 \end{aligned}$$

Check. $88.20^2 - 41.18^2 = 6083+$, and $78^2 = 6084$.

This solution should be compared with the one given in § 31, page 35. It will be seen that this is much shorter, especially as to that part in which c is found. The difference is still more marked if we remember that only part of the long division is given in § 31.

65. Given an Acute Angle and the Adjacent Side. For example, given $A = 50° 2'$, $b = 88$, find B, a, and c.



1. $B = 90° - A = 39° 58'$.

2. $\frac{a}{b} = \tan A$; $\therefore a = b \tan A$.

3. $\frac{b}{c} = \cos A$;

$\therefore b = c \cos A$, and $c = \frac{b}{\cos A}$.

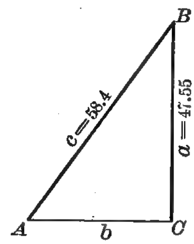
$$\begin{aligned} \log a &= \log b + \log \tan A \\ \log b &= 1.94448 \\ \log \tan A &= \underline{10.07670 - 10} \\ \log a &= 2.02118 \\ \therefore a &= 105.00 \end{aligned}$$

$$\begin{aligned} \log c &= \log b + \operatorname{colog} \cos A \\ \log b &= 1.94448 \\ \operatorname{colog} \cos A &= \underline{0.19223} \\ \log c &= 2.13671 \\ \therefore c &= 137.00 \end{aligned}$$

Check. $137^2 - 105^2 = 7744$, and $88^2 = 7744$.

This solution should be compared with the one given in § 32, page 36. Here again it will be seen that a noticeable gain is made by using logarithms, particularly in finding the value of c

66. Given the Hypotenuse and a Side. For example, given $a = 47.55$, $c = 58.4$, find A, B, and b.



1. $\sin A = \frac{a}{c}$.

2. $B = 90° - A$.

3. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.

We could, of course, find b from the equation $b = \sqrt{(c+a)(c-a)}$, as in § 33, page 36. By taking $b = a \cot A$, however, we save the trouble of first finding $c + a$ and $c - a$.

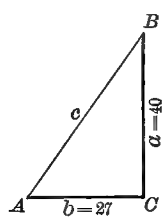
$$\begin{aligned}\log \sin A &= \log a + \operatorname{colog} c\\ \log a &= 1.67715\\ \operatorname{colog} c &= \underline{8.23359 - 10}\\ \log \sin A &= 9.91074 - 10\\ \therefore A &= 54°\ 31'\\ \therefore B &= 35°\ 29'\end{aligned}$$

$$\begin{aligned}\log b &= \log a + \log \cot A\\ \log a &= 1.67715\\ \log \cot A &= \underline{9.85300 - 10}\\ \log b &= 1.53015\\ \therefore b &= 33.896\\ &= 33.90\end{aligned}$$

Check. $58.4^2 - 33.9^2 = 2261+$, and $47.55^2 = 2261+$.

This solution should be compared with the one given in § 33, page 36.

67. Given the Two Sides. For example, given $a = 40$, $b = 27$, find A, B, and c.



1. $\tan A = \frac{a}{b}$.

2. $B = 90° - A$.

3. $\frac{a}{c} = \sin A$;

$\therefore a = c \sin A$, and $c = \frac{a}{\sin A}$.

$$\begin{aligned}\log \tan A &= \log a + \operatorname{colog} b\\ \log a &= 1.60206\\ \operatorname{colog} b &= \underline{8.56864 - 10}\\ \log \tan A &= 10.17070 - 10\\ \therefore A &= 55°\ 59'\\ \therefore B &= 34°\ 1'\end{aligned}$$

$$\begin{aligned}\log c &= \log a + \operatorname{colog} \sin A\\ \log a &= 1.60206\\ \operatorname{colog} \sin A &= \underline{0.08151}\\ \log c &= 1.68357\\ \therefore c &= 48.258\\ &= 48.26\end{aligned}$$

Check. $27^2 + 40^2 = 2329$, and $48.26^2 = 2329+$.

This solution should be compared with the solution of the same problem given in § 34, page 37. There is not much gained in this particular example because the numbers are so small that the operations are easily performed.

68. Area of a Right Triangle. The area of a triangle is equal to one half the product of the base by the altitude; therefore, if a and b denote the two sides of a right triangle and S the area, then $S = \frac{1}{2}ab$.

Hence the area may be found when a and b are known.

Consider first the case in which an acute angle and the hypotenuse are given. For example, let $A = 34° 28'$ and $c = 18.75$. Then, finding $\log a$ and $\log b$ as in § 63, we have

$$\begin{aligned} \log S &= \operatorname{colog} 2 + \log a + \log b \\ \operatorname{colog} 2 &= 9.69897 - 10 \\ \log a &= 1.02576 \\ \log b &= \underline{1.18917} \\ \log S &= 1.91390 \\ \therefore S &= 82.016 \\ &= 82.02 \end{aligned}$$

Next consider the case in which the hypotenuse and a side are given. For example, let $c = 58.4$ and $a = 47.55$. Then, finding $\log b$ as in § 66, we have

$$\begin{aligned} \log S &= \operatorname{colog} 2 + \log a + \log b \\ \operatorname{colog} 2 &= 9.69897 - 10 \\ \log a &= 1.67715 \\ \log b &= \underline{1.53015} \\ \log S &= 2.90627 \\ \therefore S &= 805.88 \\ &= 805.9 \end{aligned}$$

Finally, consider the case in which an acute angle and the opposite side are given. For example, let $A = 62° 10'$ and $a = 78$. Then, finding $\log b$ as in § 64, we have

$$\begin{aligned} \log S &= \operatorname{colog} 2 + \log a + \log b \\ \operatorname{colog} 2 &= 9.69897 - 10 \\ \log a &= 1.89209 \\ \log b &= \underline{1.61471} \\ \log S &= 3.20577 \\ \therefore S &= 1606.1 \\ &= 1606 \end{aligned}$$

We can easily verify this result, since, from § 64, $a = 78$ and $b = 41.18$; whence $\frac{1}{2}ab = 1606$, to four significant figures.

The case of an acute angle and the opposite side is treated in § 64; that of an acute angle and the adjacent side in § 65; and that of the two sides in § 67.

Exercise 31. The Right Triangle

Using logarithms, solve the following right triangles, finding the sides and areas to four figures, and the angles to minutes:

1. $a = 6$,	$c = 12$.	16. $b = 2$,	$B = 3° 38'$.
2. $b = 4$,	$A = 60°$.	17. $a = 992$,	$B = 76° 19'$.
3. $a = 3$,	$A = 30°$.	18. $a = 73$,	$B = 68° 52'$.
4. $a = 4$,	$b = 4$.	19. $a = 2.189$,	$B = 45° 25'$.
5. $a = 2$,	$c = 2.89$.	20. $b = 4$,	$A = 37° 56'$.
6. $c = 627$,	$A = 23° 30'$.	21. $c = 8590$,	$a = 4476$.
7. $c = 2280$,	$A = 28° 5'$.	22. $c = 86.53$,	$a = 71.78$.
8. $c = 72.15$,	$A = 39° 34'$.	23. $c = 9.35$,	$a = 8.49$.
9. $c = 1$,	$A = 36°$.	24. $c = 2194$,	$b = 1312.7$.
10. $c = 200$,	$B = 21° 47'$.	25. $c = 30.69$,	$b = 18.25$.
11. $c = 93.4$,	$B = 76° 25'$.	26. $a = 38.31$,	$b = 19.52$.
12. $a = 637$,	$A = 4° 35'$.	27. $a = 1.229$,	$b = 14.95$.
13. $a = 48.53$,	$A = 36° 44'$.	28. $a = 415.3$,	$b = 62.08$.
14. $a = 0.008$,	$A = 86°$.	29. $a = 13.69$,	$b = 16.92$.
15. $b = 50.94$,	$B = 43° 48'$.	30. $c = 91.92$,	$a = 2.19$.

Compute the unknown parts and also the area, having given:

31. $a = 5$,	$b = 6$.	36. $c = 68$,	$A = 69° 54'$.
32. $a = 0.615$,	$c = 70$.	37. $c = 27$,	$B = 44° 4'$.
33. $b = \sqrt[3]{2}$,	$c = \sqrt{3}$.	38. $a = 47$,	$B = 48° 49'$.
34. $a = 7$,	$A = 18° 14'$.	39. $b = 9$,	$B = 34° 44'$.
35. $b = 12$,	$A = 29° 8'$.	40. $c = 8.462$,	$B = 86° 4'$.

41. Find the value of S in terms of c and A.

42. Find the value of S in terms of a and A.

43. Find the value of S in terms of b and A.

44. Find the value of S in terms of a and c.

45. Given $S = 58$ and $a = 10$, solve the right triangle.

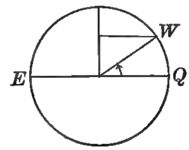
46. Given $S = 18$ and $b = 5$, solve the right triangle.

47. Given $S = 12$ and $A = 29°$, solve the right triangle.

48. Given $S = 98$ and $c = 22$, solve the right triangle.

49. Find the two acute angles of a right triangle if the hypotenuse is equal to three times one of the sides.

50. The latitude of Washington is 38° 55′ 15″ N. Taking the radius of the earth as 4000 mi., what is the radius of the circle of latitude of Washington? What is the circumference of this circle?

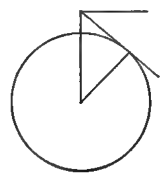


In all such examples the earth will be considered as a perfect sphere with the radius as above given, unless the contrary is stated. For more accurate data consult the Table of Constants.

51. What is the difference between the length of a degree of latitude and the length of a degree of longitude at Washington?

Use the data given in Ex. 50.

52. From the top of a mountain 1 mi. high, overlooking the sea, an observer looks toward the horizon. What is the angle of depression of the line of sight?



In the figure the height of the mountain is necessarily exaggerated. The angle is so small that the result can be found by five-place tables only between two limits which differ by 3′ 40″.

53. At a horizontal distance of 120 ft. from the foot of a steeple, the angle of elevation of the top is found to be 60° 30′. Find the height of the steeple above the instrument.

54. From the top of a rock which rises vertically 326 ft. out of the water, the angle of depression of a boat is found to be 24°. Find the distance of the boat from the base of the rock.

55. How far from the eye is a monument on a level plain if the height of the monument is 200 ft. and the angle of elevation of the top is 3° 30′?

56. A distance AB of 96 ft. is measured along the bank of a river from a point A opposite a tree C on the other bank. The angle ABC is 21° 14′. Find the breadth of the river.

57. What is the angle of elevation of an inclined plane if it rises 1 ft. in a horizontal distance of 40 ft.?

58. Find the angle of elevation of the sun when a tower 120 ft. high casts a horizontal shadow 70 ft. long.

59. How high is a tree which casts a horizontal shadow 80 ft. in length when the angle of elevation of the sun is 50°?

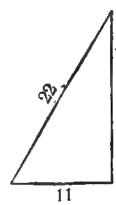
60. A rectangle 7.5 in. long has a diagonal 8.2 in. long. What angle does the diagonal make with the base?

61. A rectangle $8\frac{1}{4}$ in. long has an area of $49\frac{1}{2}$ sq. in. Find the angle which the diagonal makes with the base.

62. The length AB of a rectangular field $ABCD$ is 80 rd. and the width AD is 60 rd. The field is divided into two equal parts by a straight fence PQ starting from a point P on AD which is 15 rd. from A. What angle does PQ make with AD?

63. A ship is sailing due northeast at the rate of 10 mi. an hour. Find the rate at which she is moving due north, and also due east.

64. If the foot of a ladder 22 ft. long is 11 ft. from a house, how far up the side of the house does the ladder reach?

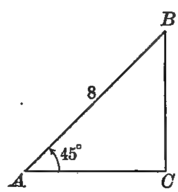


65. In front of a window 20 ft. from the ground there is a flower bed 6 ft. wide and close to the house. How long is a ladder which will just reach from the outside edge of the bed to the window?

66. A ladder 40 ft. long can be so placed that it will reach a window 33 ft. above the ground on one side of the street, and by tipping it back without moving its foot it will reach a window 21 ft. above the ground on the other side. Find the width of the street.

67. From the top of a hill the angles of depression of two successive milestones, on a straight, level road leading to the hill, are 5° and 15°. Find the height of the hill.

68. A stick 8 ft. long makes an angle of 45° with the floor of a room, the other end resting against the wall. How far is the foot of the stick from the wall?



69. A building stands on a horizontal plain. The angle of elevation at a certain point on the plain is 30°, and at a point 100 ft. nearer the building it is 45°. How high is the building?

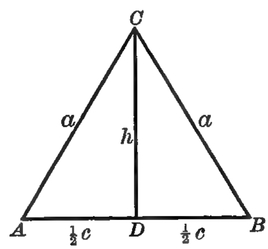
70. From a certain point on the ground the angles of elevation of the top of the belfry of a church and of the top of the steeple are found to be 40° and 51° respectively. From a point 300 ft. further off, on a horizontal line, the angle of elevation of the top of the steeple is found to be 33° 45′. Find the height of the top of the steeple above the top of the belfry.

71. The angle of elevation of the top C of an inaccessible fort observed from a point A is 12°. At a point B, 219 ft. from A and on a line AB perpendicular to AC, the angle ABC is 61° 45′. Find the height of the fort.

69. The Isosceles Triangle. Since an isosceles triangle is divided by the perpendicular from the vertex to the base into two congruent right triangles, an isosceles triangle is determined by any two parts which determine one of these right triangles.

In the examples which follow we shall represent the parts of the isosceles triangle ABC, among which the altitude CD is included, as follows:

$a =$ one of the equal sides,
$c =$ the base,
$h =$ the altitude,
$A =$ one of the equal angles,
$C =$ the angle at the vertex.



For example, given a and c, find A, C, and h.

1. $\cos A = \dfrac{\frac{1}{2}c}{a} = \dfrac{c}{2a}.$

2. $C + 2A = 180°; \therefore C = 180° - 2A = 2(90° - A).$

3. h may be found by any one of the following equations:

$$h^2 + \tfrac{1}{4}c^2 = a^2,$$

whence $$h = \sqrt{(a + \tfrac{1}{2}c)(a - \tfrac{1}{2}c)};$$

or $$\frac{h}{a} = \sin A, \text{ whence } h = a \sin A;$$

or $$\frac{h}{\frac{1}{2}c} = \tan A, \text{ whence } h = \tfrac{1}{2}c \tan A.$$

When c and h are known, the area can be found by the formula

$$S = \tfrac{1}{2}ch$$

That is, $$S = \tfrac{1}{2}c \cdot a \sin A = \tfrac{1}{2}ac \sin A,$$

or $$S = \tfrac{1}{2}c \cdot \tfrac{1}{2}c \tan A = \tfrac{1}{4}c^2 \tan A,$$

or $$S = \tfrac{1}{2}c\sqrt{(a + \tfrac{1}{2}c)(a - \tfrac{1}{2}c)}.$$

Consider also the case in which a and h are given, to find A, C, c, and S.

1. $\sin A = \dfrac{h}{a}$, and hence A is known.

2. $C = 2(90° - A)$, as above, and hence C is known.

3. $\frac{1}{2}c = a \cos A$, and hence c is known.

4. $S = \frac{1}{2}ch$, and hence S is known.

We can also find S from any of its other equivalents, such as those given above, or $a^2 \sin \frac{1}{2}C \sin A$, each of which is easily deduced.

Exercise 32. The Isosceles Triangle

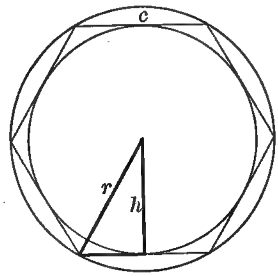
Solve the following isosceles triangles:

1. Given a and A, find C, c, and h.
2. Given a and C, find A, c, and h.
3. Given c and A, find C, a, and h.
4. Given c and C, find A, a, and h.
5. Given h and A, find C, a, and c.
6. Given h and C, find A, a, and c.
7. Given a and h, find A, C, and c.
8. Given c and h, find A, C, and a.
9. Given $a = 14.3$, $c = 11$, find A, C, and h.
10. Given $a = 0.295$, $A = 68° 10'$, find c, h, and S.
11. Given $c = 2.352$, $C = 69° 49'$, find a, h, and S.
12. Given $h = 7.4847$, $A = 76° 14'$, find a, c, and S.
13. Given $c = 147$, $S = 2572.5$, find A, C, a, and h.
14. Given $h = 16.8$, $S = 43.68$, find A, C, a, and c.
15. Given $a = 27.56$, $A = 75° 14'$, find c, h, and S.

Given an isosceles triangle, ABC:

16. Find the value of S in terms of a and C.
17. Find the value of S in terms of a and A.
18. Find the value of S in terms of h and C.
19. A barn is 40 ft. by 80 ft., the pitch of the roof is 45°; find the length of the rafters and the area of the whole roof.
20. In a unit circle what is the length of the chord subtending the angle 45° at the center?
21. The radius of a circle is 30 in., and the length of a chord is 44 in.; find the angle subtended at the center.
22. Find the radius of a circle if a chord whose length is 5 in. subtends at the center an angle of 133°.
23. What is the angle at the center of a circle if the subtending chord is equal to $\frac{2}{3}$ of the radius?
24. Find the area of a circular sector if the radius of the circle is 12 in., and the angle of the sector is 30°.
25. Find the tangent of the angle of the slope of an A-roof of a building which is 24 ft. 6 in. wide at the eaves, the ridgepole being 10 ft. 9 in. above the eaves.

70. The Regular Polygon. We have already considered a few cases involving the regular polygon. It is evident from geometry that if the polygon shown below has n sides, the angle of the right triangle which has its vertex at the center is equal to $\frac{1}{2}$ of $360°/n$, or $180°/n$. The triangle may evidently be solved if the radius of the circumscribed circle (r), the radius of the inscribed circle (h), or the side of the polygon (c) is given.



In the exercises we shall let

$n =$ number of sides,
$c =$ length of one side,
$r =$ radius of circumscribed circle,
$h =$ radius of inscribed circle,
$p =$ the perimeter,
$S =$ the area.

Then, by geometry,

$$S = \tfrac{1}{2}\,hp.$$

Exercise 33. The Regular Polygon

Find the remaining parts of a regular polygon, given:

1. $n = 10,\ c = 1.$	3. $n = 20,\ r = 20.$	5. $n = 11,\ S = 20.$
2. $n = 18,\ r = 1.$	4. $n = 8,\ h = 1.$	6. $n = 7,\ S = 7.$

7. The side of a regular inscribed hexagon is 1 in.; find the side of a regular inscribed dodecagon.

8. Given n and c, and represent by b the side of the regular inscribed polygon having $2n$ sides, find b in terms of n and c.

9. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16 in.

10. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12 sq. in.

11. Find the perimeter of a regular dodecagon circumscribed about a circle the circumference of which is 1 in.

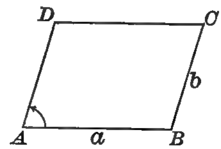
12. What is the side of the regular inscribed polygon of 100 sides, the radius of the circle being unity? What is the perimeter?

13. What is the perimeter of the regular inscribed polygon of 360 sides, the radius of the circle being unity?

14. The area of a regular polygon of twenty-five sides is 40 sq. in; find the area of the ring included between the circumferences of the inscribed and circumscribed circles.

Exercise 34. Review Problems

1. Prove that the area of the parallelogram here shown is equal to $ab \sin A$.



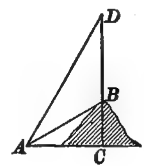
2. Two sides of a parallelogram are 5 in. and 6 in. respectively, and their included angle is 82° 45′. What is the area?

3. Two sides of a parallelogram are 9 ft. and 12 ft. respectively, and their included angle is 74.5°. What is the area?

4. Each side of a rhombus is 7.35 in., and one angle is 42° 27′. What is the area?

5. The area of a rhombus is 250 sq. in., and one of the angles is 37° 25′. What is the length of each side?

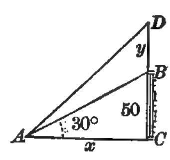
6. A pole BD stands on the top of a mound BC. From a point A the angles of elevation of the top and foot of the pole are 60° and 30° respectively. Prove that the height of the pole is twice the height of the mound.



7. A ladder 38 ft. long is resting against a wall. The foot of the ladder is 7 ft. 2 in. from the wall. What is the height of the top of the ladder above the ground?

8. From a boat 1325 ft. from the base of a vertical cliff the angle of elevation of the top of the cliff is observed to be 14° 30′. Find the height of the cliff.

9. On the top of a building 50 ft. high there is a flagstaff BD. From a point A on the ground the angles of elevation of B and D are 30° and 45° respectively. Find the length of the flagstaff and the distance AC of the observer from the building, as shown in the annexed figure.

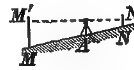


Since $\frac{50}{x} = \tan 30°$ and $\frac{50+y}{x} = \tan 45°$, x can evidently be eliminated.

10. A man whose eye is 5 ft. 8 in. above the ground stands midway between two telegraph poles which are 200 ft. apart. The elevation of the top of each pole is 48° 50′. What is the height of each?

11. The captain of a ship observed a lighthouse directly to the east. After sailing north 2 mi. he observed it to lie 55° 30′ east of south. How far was the ship from the lighthouse at the time of each observation?

12. A leveling instrument is placed at A on the slope MN, and the line $M'N'$ is sighted to two upright rods. By measurement MM' is found to be 12.8 ft., NN' to be 3.4 ft., and $M'N'$ to be 48.3 ft. Required the angle of the slope of MN and the distance MN.

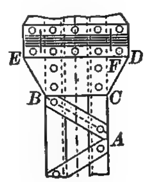


13. A wire stay is fastened to a telegraph pole 6.8 ft. from the ground and is stretched tightly so as to reach the ground 5.2 ft. from the foot of the pole. What angle does the wire stay make with the ground?

14. The top of a conical tent is 8 ft. 7 in. above the ground, and the diameter of the base is 9 ft. 8 in. Find the inclination of the side of the tent to the horizontal. Check the result by drawing the figure to scale and measuring the angle with a protractor.

15. In this piece of iron construction work $BC = 11$ in. and AB makes an angle of 30° with BC. What is the length of AC?

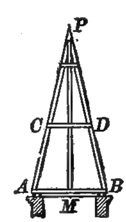
16. In Ex. 15 it is also known that BE and CD are each 9 in. long and make angles of 60° with BC produced. What is the length of ED?



17. From the conditions given in Ex. 16, find the length of CF.

18. The base of a rectangle is $14\frac{5}{8}$ in. and the diagonal is $19\frac{1}{8}$ in. What angle does the diagonal make with the base? Check the result by drawing the figure to scale and measuring the angle with a protractor.

19. In constructing the spire represented in the figure below it is planned to have $AB = 42$ ft. and $PM = 92$ ft. What angle of slope must the builders give to AP?



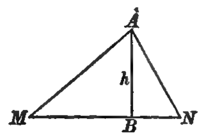
20. In Ex. 19 find the length of AP and find the angle P.

21. In the figure of Ex. 19 the brace CD is put in 38 ft. above AB. What is its length?

22. The spire of Ex. 19 rests on a tower. A man standing on the ground at a distance of 400 ft. from the base of the tower observes the angle of elevation of P to be 25° 38′, the instrument being 5 ft. above the ground. What is the height of P above the ground?

23. When the angle of elevation of the sun is 38.4°, what is the length of the shadow of a tower 175 ft. high?

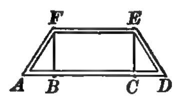
24. Two men, M and N, 3200 ft. apart, observe an aeroplane A at the same instant, and at a time when the plane MNA is vertical. The angle of elevation at M is $41° 27'$ and the angle at N is $61° 42'$. Required AB, the height of the aeroplane.



Show that $h \cot 41° 27' + h \cot 61° 42'$ is known, whence h can be found.

25. A kite string 475 ft. long makes an angle of elevation of $49° 40'$. Assuming the string to be straight, what is the altitude of the kite?

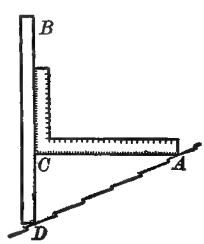
26. A steel bridge has a truss $ADEF$ in which it is given that $AD = 20$ ft., $BF = 6$ ft. 8 in., and $FE = 12$ ft., as shown in the figure. Required the angle of slope which AF makes with AD.



27. Two tangents are drawn from a point P to a circle and contain an angle of $37.4°$. The radius of the circle is 5 in. Find the length of each tangent and the distance of P from the center.

28. From the top of a cliff 95 ft. high, the angles of depression of two boats at sea are observed, by the aid of an instrument 5 ft. above the ground, to be 45° and 30° respectively. The boats are in a straight line with a point at the foot of the cliff directly beneath the observer. What is the distance between the boats?

29. A carpenter's square BCA is held against the vertical stick BD resting on a sloping roof AD, as in the figure. It is found that $AC = 24$ in. and $CD = 11.5$ in. Find the angle of slope of the roof with the horizontal.



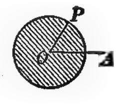
30. In Ex. 29 find the length of AD.

31. A man 6 ft. tall stands 4 ft. 9 in. from a street lamp. If the length of his shadow is 19 ft., how high is the light above the street?

32. The shadow of a city building is observed to be 100 ft. long, and at the same time the shadow of a lamp-post 9 ft. high is observed to be 5.2 ft. long. Find the angle of elevation of the sun and the height of the building.

33. A man 5 ft. 10 in. tall walks along a straight line that passes at a distance of 2 ft. 9 in. from a street light. If the light is 9 ft. 6 in. above the ground, find the length of the man's shadow when his distance from the point on his path that is nearest to the lamp is 3 ft. 8 in.

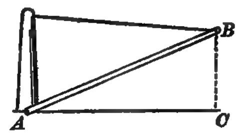
34. A man on a bridge 35 ft. above a stream, using an instrument 5 ft. high, sees a rowboat at an angle of depression of 27° 30′. If the boat is approaching at the rate of $2\frac{3}{4}$ mi. an hour, in how many seconds will it reach the bridge?



35. A shaft O, of diameter 4 in., makes 480 revolutions per minute. If the point P starts on the horizontal line OA, how far is it above OA after $\frac{1}{48}$ of a second?

36. Assuming the earth to be a sphere with radius 3957 mi., find the radius of the circle of latitude which passes through a place in latitude 47° 27′ 10″ N.

37. When a hoisting crane AB, 28 ft. long, makes an angle of 23° with the horizontal AC, what is the length of AC? Suppose that the angle CAB is doubled, what is then the length of AC?



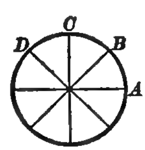
38. In Ex. 37 find the length of BC in each of the two cases.

39. Wishing to measure the distance AB, a man swings a 100-foot tape line about B, describing an arc on the ground, and then does the same about A. The arcs intersect at C, and the angle ACB is found to be 32° 10′. What is the length of AB?

40. From the top of a mountain 15,250 ft. high, overlooking the sea to the south, over how many minutes of latitude can a person see if he looks southward? Use the assumption stated in Ex. 36.

41. The length of each blade of a pair of shears, from the screw to the point, is $5\frac{1}{4}$ in. When the points of the open shears are $3\frac{7}{8}$ in. apart, what angle do the blades make with each other?

42. In Ex. 41 how far apart are the points when the blades make an angle of 28° 45′ with each other?



43. The wheel here represented has eight spokes, each being 19 in. long. How far is it from A to B? from B to D?

44. The angle of elevation of a balloon from a station directly south of it is 60°. From a second station lying 5280 ft. directly west of the first one the angle of elevation is 45°. The instrument being 5 ft. above the level of the ground, what is the height of the balloon?

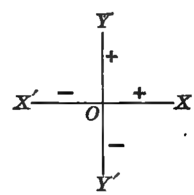
CHAPTER V

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

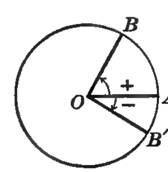
71. Need for Oblique Angles. We have thus far considered only right triangles, or triangles which can readily be cut into right triangles for purposes of solution. There are, however, oblique triangles which cannot conveniently be solved by merely separating them into right triangles. We are therefore led to consider the functions of oblique angles, and to enlarge our idea of angles so as to include angles greater than 180°, angles greater than 360°, and even negative angles and the angle 0°.

72. Positive and Negative Angles. We have learned in algebra that we may distinguish between two lines which extend in opposite directions by calling one *positive* and the other *negative*.

For example, in the annexed figure we consider OX as positive and therefore OX' as negative. We also consider OY as positive and hence OY' as negative. In general, horizontal lines extending to the right of a point which we select as zero are considered positive, and those to the left negative. Vertical lines extending upward from zero are considered positive, and those extending downward are considered negative.



With respect to angles, an angle is considered *positive* if the rotating line which describes it moves counterclockwise, that is, in the direction opposite to that taken by the hands of a clock. An angle is considered *negative* if the rotating line moves clockwise, that is, in the same direction as that taken by the hands of a clock.



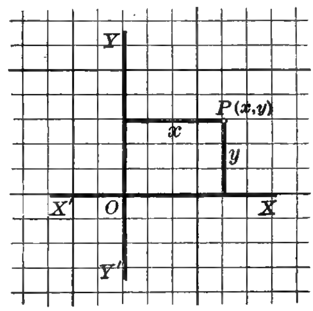
Arcs which subtend positive angles are considered *positive*, and arcs which subtend negative angles are considered *negative*. Thus $\angle AOB$ and arc AB are considered positive; $\angle AOB'$ and arc AB' are considered negative.

For example, we may think of a pendulum as swinging through a positive angle when it swings to the right, and through a negative angle when it swings to the left. We may also think of an angle of elevation as positive and an angle of depression as negative, if it appears to be advantageous to do so in the solution of a problem.

73. Coördinates of a Point. In trigonometry, as in work with graphs in algebra, we locate a point in a plane by means of its distances from two perpendicular lines.

These lines are lettered XX' and YY', and their point of intersection O. The lines are called the *axes* and the point of intersection the *origin*.

In some branches of mathematics it is more convenient to use oblique axes, but in trigonometry rectangular axes are used as here shown.



The distance of any point P from the axis XX', or the x-axis, is called the *ordinate* of the point. Its distance from the axis YY', or the y-axis, is called the *abscissa* of the point.

In the figure, y is the ordinate of P, and x is the abscissa of P. The point P is represented by the symbol (x, y). In the figure the side of each small square may be taken to represent one unit, in which case $P = (4, 3)$, because its abscissa is 4 and its ordinate 3. Following a helpful European custom, the points are indicated by small circles, so as to show more clearly when a line is drawn through them.

The abscissa and ordinate of a point are together called the *coördinates of the point.*

74. Signs of the Coördinates. From § 73 we see that *abscissas to the right of the y-axis are positive; abscissas to the left of the y-axis are negative; ordinates above the x-axis are positive; ordinates below the x-axis are negative.*

A point on the line YY' has zero for its abscissa, and hence the abscissa may be considered as either positive or negative and may be indicated by ± 0. Similarly, a point on the line XX' has ± 0 for its ordinate.

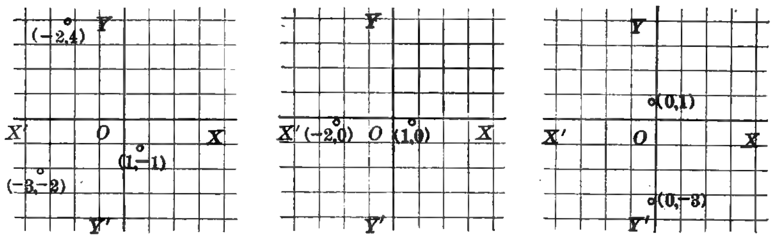
75. The Four Quadrants. The axes divide the plane into four parts known as *quadrants.*

Because angles are generally considered as generated by the rotating line moving counterclockwise, the four quadrants are named in a counterclockwise order. Quadrant XOY is spoken of as the first quadrant, YOX' as the second quadrant, $X'OY'$ as the third quadrant, and $Y'OX$ as the fourth quadrant.

76. Signs of the Coördinates in the Several Quadrants. From § 74 we have the following rule of signs:

In quadrant I the abscissa is positive, the ordinate positive;
In quadrant II the abscissa is negative, the ordinate positive;
In quadrant III the abscissa is negative, the ordinate negative;
In quadrant IV the abscissa is positive, the ordinate negative.

77. Plotting a Point. Locating a point, having given its coördinates, is called *plotting the point.*



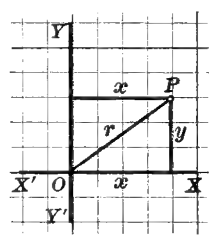
For example, in the first of these figures the point $(-2, 4)$ is shown in quadrant II, the point $(-3, -2)$ in quadrant III, and the point $(1, -1)$ in quadrant IV.

In the second figure the point $(-2, 0)$ is shown on OX', between quadrants II and III, and the point $(1, 0)$ on OX, between quadrants I and IV.

In the third figure the point $(0, 1)$ is shown on OY, between quadrants I and II, and the point $(0, -3)$ on OY', between quadrants III and IV.

In every case the origin O may be designated as the point $(0,0)$.

78. Distance from the Origin. The coördinates of P being x and y, we may form a right triangle the hypotenuse of which is the distance of P from O.



Representing OP by r, we have

$$r = \sqrt{x^2 + y^2}.$$

Since this may be written $r = \pm\sqrt{x^2+y^2}$, we see that r may be considered as either positive or negative. It is the custom, however, to consider the rotating line which forms the angle as positive. If r is produced through O, the production is considered as negative.

1. What is the distance of the point $(3, 4)$ from the origin?

$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

2. What is the distance of the point $(-3, -2)$ from the origin?

$$r = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13} = 3.61.$$

3. What is the distance of the point $(5, -5)$ from the origin?

$$r = \sqrt{5^2 + (-5)^2} = \sqrt{50} = 7.07.$$

4. What is the distance of the point $(-2, 0)$ from the origin?

$$r = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2,$$

as is evident from the conditions of the problem.

Exercise 35. Distances from the Origin

Using squared paper, or measuring with a ruler, plot the following points:

1. $(2, 3)$.	8. $(-3, 2)$.	15. $(3, -4)$.	22. $(0, 0)$.
2. $(3, 5)$.	9. $(-3, 4)$.	16. $(4, -3)$.	23. $(0, 2\frac{1}{2})$.
3. $(4, 4)$.	10. $(-5, 1)$.	17. $(5, -1)$.	24. $(0, -3\frac{1}{2})$.
4. $(2\frac{1}{2}, 3)$.	11. $(-4, 6)$.	18. $(0, 7)$.	25. $(4\frac{1}{2}, 0)$.
5. $(3\frac{1}{2}, 4\frac{1}{2})$.	12. $(-2, -2)$.	19. $(3, 0)$.	26. $(5\frac{1}{2}, 0)$.
6. $(4\frac{1}{4}, 4\frac{1}{4})$.	13. $(-3, -5)$.	20. $(0, -4)$.	27. $(-2\frac{1}{2}, 0)$.
7. $(5\frac{1}{2}, 3\frac{1}{2})$.	14. $(-5, -3)$.	21. $(-2, 0)$.	28. $(-3\frac{1}{4}, 0)$.

Find the distance of each of the following points from the origin:

29. $(6, 8)$.	32. $(1\frac{1}{2}, 2)$.	35. $(2, \sqrt{5})$.	38. $(0, 7)$.
30. $(9, 12)$.	33. $(\frac{3}{4}, 1)$.	36. $(-3, 4)$.	39. $(5, 0)$.
31. $(5, 12)$.	34. $(2\frac{1}{4}, 3)$.	37. $(0, 0)$.	40. $(-12, -9)$.

41. Find the distance from $(3, 2)$ to $(-2, 3)$.

42. Find the distance from $(-3, -2)$ to $(2, -3)$.

43. Find the distance from $(4, 1)$ to $(-4, -1)$.

44. Find the distance from $(0, 3)$ to $(-3, 0)$.

45. A point moves to the right 7 in., up 4 in., to the right 10 in., and up $18\frac{2}{3}$ in. How far is it then from the starting point?

46. A point moves to the right 9 in., up 5 in., to the left 4 in., and up 3 in. How far is it then from the starting point?

47. Find the distance from $\left(-\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$ to $\left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$.

48. A triangle is formed by joining the points $(1, 0)$, $\left(-\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$, and $\left(-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$. Find the perimeter of the triangle. Draw the figure to scale.

49. Find the area of the triangle in Ex. 48.

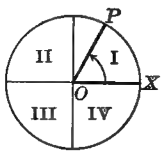
50. A hexagon is formed by joining in order the points $(1, 0)$, $\left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$, $\left(-\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$, $(-1, 0)$, $\left(-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$, and $(1, 0)$. Is the figure a regular hexagon? Prove it.

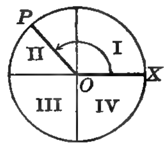
51. A polygon is formed by joining in order the points $(1, 0)$, $\left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$, $(0, 1)$, $\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$, $(-1, 0)$, $\left(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right)$, $(0, -1)$, $\left(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right)$, and $(1, 0)$. Draw the figure, state the kind of polygon, and find its area.

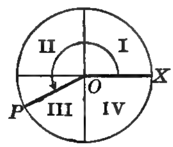
79. Angles of any Magnitude. In the following figures, if the rotating line OP revolves about O from the position OX, in a counterclockwise direction, until it again coincides with OX, it will generate all angles in every quadrant from 0° to 360°.

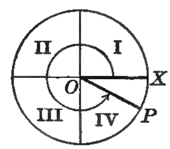
The line OX is called the *initial side* of the angle, and the line OP the *terminal side* of the angle.

An angle is said to be an angle of that quadrant in which its terminal side lies.



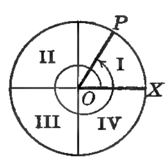


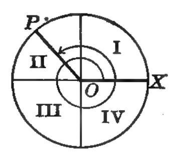




Angles between 0° and 90° are angles of quadrant I.
Angles between 90° and 180° are angles of quadrant II.
Angles between 180° and 270° are angles of quadrant III.
Angles between 270° and 360° are angles of quadrant IV.

The rotating line may also pass through 360°, forming angles from 360° to 720°. It may then make another revolution, forming angles greater than 720°, and so on indefinitely.





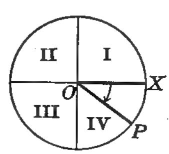
For example, in using a screwdriver we turn through angles of 360°, 720°, 1080°, and so on, depending upon the number of revolutions. In the same way, the minute hand of a clock turns through 8640° in a day, and the drive wheel of an engine may turn through thousands of degrees in an hour.

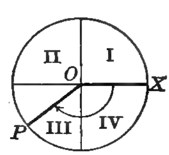
We might, if necessary, speak of an angle of 400° as an angle of quadrant I, because its terminal side is in that quadrant, but we have no occasion to do so in practical cases.

As stated in § 72, if the line OP is rotated clockwise, it generates negative angles.

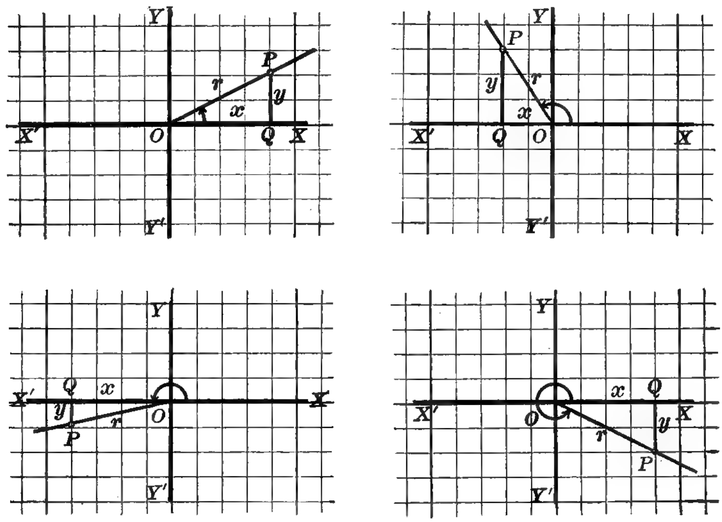
In this way we may form angles of − 40° or − 140°, as here shown, and the rotation may continue until we have angles of − 360°, − 720°, − 1080°, − 1440°, and so on indefinitely.

We shall have but little need for the negative angle in the practical work of trigonometry, but we shall make extensive use of angles between 0° and 180°, and some use of those between 180° and 360°.





80. Functions of Any Angle. Since we have now seen that we may have angles of any magnitude, it is necessary to consider their functions. Although we must define these functions anew, it will be seen that the definitions hold for the acute angles which we have already considered.



In whatever quadrant the angle is, we designate it by A. We take a point P, or (x, y), on the rotating line, and let $OP = r$. Then the angle XOP, read counterclockwise, is the angle A. We then define the functions as follows:

$$\sin A = \frac{y}{r} = \frac{\text{ordinate}}{\text{distance}}, \qquad \csc A = \frac{1}{\sin A} = \frac{r}{y} = \frac{\text{distance}}{\text{ordinate}},$$

$$\cos A = \frac{x}{r} = \frac{\text{abscissa}}{\text{distance}}, \qquad \sec A = \frac{1}{\cos A} = \frac{r}{x} = \frac{\text{distance}}{\text{abscissa}},$$

$$\tan A = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}, \qquad \cot A = \frac{1}{\tan A} = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}.$$

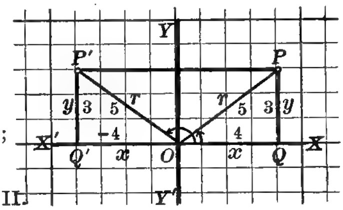
It will be seen that these definitions are practically the same as those already learned for angles in quadrant I. Their application to the other quadrants is apparent. The general definitions might have been given at first, but this plan offers difficulties for a beginner which make it undesirable.

By counting the squares on squared paper and thus getting the lengths of certain lines, the approximate values of the functions of any given angle may be found, but the exercise has no practical significance. The values of the functions are determined by series, these being explained in works on the calculus.

81. Angles determined by Functions. Given any function of an angle, it is possible to construct the angle or angles which satisfy the value of the function.

1. Given $\sin A = \frac{3}{5}$, construct the angle A.

If we take a line parallel to $X'X$ and 3 units above it, and then rotate a line OP, 5 units long, about O until P rests upon this parallel, we shall have in the annexed figure



$$OP = 5,\ PQ = 3,$$

and likewise $$OP' = 5,\ P'Q' = 3.$$

Then $\sin A = \dfrac{3}{5} = \dfrac{y}{r} = \dfrac{PQ}{OP}$, in quadrant I;

and $\sin A' = \dfrac{3}{5} = \dfrac{y}{r} = \dfrac{P'Q'}{OP'}$, in quadrant II.

In other words, we have constructed two angles, each of which has $\frac{3}{5}$ for its sine.

Furthermore, we could construct an infinite number of such angles, for we see that $360° + A$ terminates in OP and has the same sine that A has, and that the same may be said of $360° + A'$, $720° + A$, $720° + A'$, $1080° + A$, and so on.

In general, therefore, the angle $n \times 360° + A$ has the same functions as A, n being any integer. Hence if we know the value of any particular function of an angle, the angle cannot be uniquely determined; that is, there is more than one angle which satisfies the condition. In general, as we see, an infinite number of angles will satisfy the given condition, although this gives no trouble because only two of these angles can be less than 360°.

2. Given $\tan A = \frac{3}{4}$, construct the angle A.

If we take an abscissa 4 and an ordinate 3, as in quadrant I of the figure, we locate the point (3, 4). Then angle XOP has for its tangent $\frac{3}{4}$. But it is evident that we may also locate the point $(-3, -4)$ in quadrant III, and thus find an angle between 180° and 270° whose tangent is $\frac{3}{4}$.

82. Functions found from Other Functions. Given any function of an angle, it is possible not only to construct the angle but also to find the other functions.

For in Ex. 1 above, after constructing angles A and A', we see that

$$\sin A = \frac{3}{5}, \qquad \csc A = \frac{5}{3},$$

$$\cos A = \frac{4}{5} \text{ or } \frac{-4}{5}, \qquad \sec A = \frac{5}{4} \text{ or } \frac{5}{-4},$$

$$\tan A = \frac{3}{4} \text{ or } \frac{3}{-4}, \qquad \cot A = \frac{4}{3} \text{ or } \frac{-4}{3}.$$

That is, if $\sin A = \frac{3}{5}$, then $\cos A = \pm\frac{4}{5}$, $\tan A = \pm\frac{3}{4}$, $\csc A = \frac{5}{3}$, $\sec A = \pm\frac{5}{4}$, and $\cot A = \pm\frac{4}{3}$.

Exercise 36. Construction of Angles and Functions

Using the protractor, construct the following angles:

1. $30°$.	4. $150°$.	7. $270°$.	10. $405°$.	13. $-45°$.
2. $60°$.	5. $180°$.	8. $300°$.	11. $450°$.	14. $-90°$.
3. $80°$.	6. $200°$.	9. $360°$.	12. $720°$.	15. $-180°$.

State the quadrants in which the terminal sides of the following angles lie:

16. $45°$.	19. $150°$.	22. $390°$.	25. $660°$.	28. $930°$.
17. $75°$.	20. $210°$.	23. $495°$.	26. $765°$.	29. $990°$.
18. $120°$.	21. $315°$.	24. $570°$.	27. $820°$.	30. $1080°$.

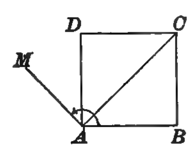
Construct two angles A, given the following:

31. $\sin A = \frac{1}{2}$.	36. $\sin A = -\frac{3}{4}$.	41. $\sin A = -1$.
32. $\cos A = \frac{1}{2}$.	37. $\cos A = -\frac{4}{5}$.	42. $\cos A = -1$.
33. $\tan A = \frac{1}{2}$.	38. $\tan A = -\frac{2}{3}$.	43. $\tan A = -1$.
34. $\cot A = \frac{1}{2}$.	39. $\cot A = -\frac{4}{5}$.	44. $\cot A = -1$.
35. $\sec A = 2$.	40. $\sec A = -1$.	45. $\sec A = -2$.

Given the following functions of angle A, construct the other functions:

46. $\sin A = \frac{2}{3}$.	51. $\sin A = -\frac{4}{5}$.	56. $\sin A = -\frac{1}{2}$.
47. $\cos A = \frac{3}{4}$.	52. $\cos A = -1$.	57. $\cos A = -\frac{1}{2}$.
48. $\tan A = \frac{4}{5}$.	53. $\tan A = -\frac{3}{8}$.	58. $\tan A = -\frac{1}{2}$.
49. $\cot A = \frac{3}{8}$.	54. $\sec A = -2$.	59. $\cot A = -\frac{1}{2}$.
50. $\csc A = 3$.	55. $\csc A = -1$.	60. $\sec A = -2\frac{1}{2}$.

61. If $\tan A = \sqrt{2}$, show that $\cot A$ is half as large. What are the values of $\sin A$, $\cos A$, $\sec A$, and $\csc A$?



62. The product $2 \sin 45° \cos 45°$ is equal to the sine of what angle?

63. The product $2 \sin 30° \cos 30°$ is equal to the sine of what angle?

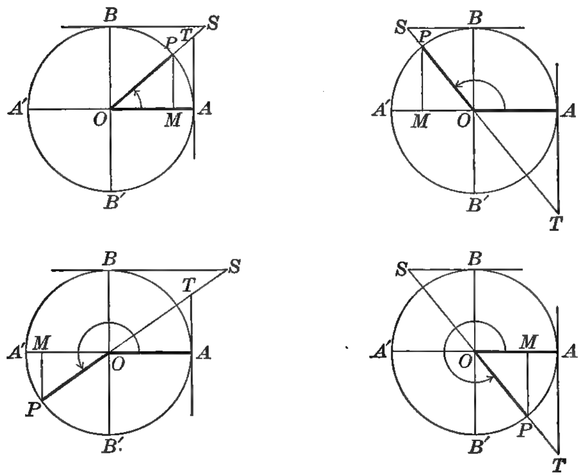
64. To the diagonal AC of a square $ABCD$ a perpendicular AM is drawn. Find the values of the six functions of angle BAM.

65. In the figure of Ex. 64, suppose AM rotates further, until it is in line with BA. What are then the six functions of angle BAM?

83. Line Values of the Functions. As in the case of acute angles (§ 22) we may represent the trigonometric functions of any angle by means of lines in a circle of radius unity.

Thus in each of the following figures

$$\sin x = MP, \qquad \tan x = AT, \qquad \sec x = OT,$$
$$\cos x = OM, \qquad \cot x = BS, \qquad \csc x = OS.$$



By examining the figures we see that

In quadrant I *all* the functions are positive;
In quadrant II the *sine and cosecant only* are positive;
In quadrant III the *tangent and cotangent only* are positive;
In quadrant IV the *cosine and secant only* are positive.

It will be seen as we proceed that the laws and relations which have been found for the functions of acute angles hold for the functions of angles greater than 90°. For example, it is apparent from the above figures that, in every quadrant,

$$\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2 = 1,$$

and hence that $\qquad \sin^2 A + \cos^2 A = 1,$

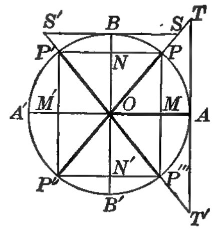
as shown in § 14. It is also evident that

$$\frac{AT}{1} = \frac{MP}{OM},$$

and hence that $\qquad \tan A = \dfrac{\sin A}{\cos A}.$

Other similar relations are easily proved by reference to the figures.

84. Variations in the Functions. A study of the line values of the functions shows how they change as the angle increases from 0° to 360°.



1. *The Sine.* In the first quadrant the sine MP is positive, and increases from 0 to 1; in the second it remains positive, and decreases from 1 to 0; in the third it is negative, and increases in absolute value from 0 to 1; in the fourth it is negative, and decreases in absolute value from 1 to 0. The absolute value of the sine varies, therefore, from 0 to 1, and its total range of values is from $+1$ to -1.

In the third quadrant the sine *decreases* from 0 to -1, but the *absolute value* (the value without reference to its sign) *increases* from 0 to 1, and similarly for other cases on this page in which the absolute value is mentioned.

2. *The Cosine.* In the first quadrant the cosine OM is positive, and decreases from 1 to 0; in the second it becomes negative, and increases in absolute value from 0 to 1; in the third it is negative, and decreases in absolute value from 1 to 0; in the fourth it is positive, and increases from 0 to 1. The absolute value of the cosine varies, therefore, from 0 to 1.

3. *The Tangent.* In the first quadrant the tangent AT is positive, and increases from 0 to ∞; in the second it becomes negative, and decreases in absolute value from ∞ to 0; in the third it is positive, and increases from 0 to ∞; in the fourth it is negative, and decreases in absolute value from ∞ to 0.

4. *The Cotangent.* In the first quadrant the cotangent BS is positive, and decreases from ∞ to 0; in the second it is negative, and increases in absolute value from 0 to ∞; in the third and fourth quadrants it has the same sign, and undergoes the same changes as in the first and second quadrants respectively. The tangent and cotangent may therefore have any values whatever, positive or negative.

5. *The Secant.* In the first quadrant the secant OT is positive, and increases from 1 to ∞; in the second it is negative, and decreases in absolute value from ∞ to 1; in the third it is negative, and increases in absolute value from 1 to ∞; in the fourth it is positive, and decreases from ∞ to 1.

6. *The Cosecant.* In the first quadrant the cosecant OS is positive, and decreases from ∞ to 1; in the second it is positive, and increases from 1 to ∞; in the third it is negative, and decreases in absolute value from ∞ to 1; in the fourth it is negative, and increases in absolute value from 1 to ∞.

It is evident, therefore, that the sine can never be greater than 1 nor less than -1, and that it has these limiting values at 90° and 270° respectively. We may also say that its absolute value can never be greater than 1, and that it has its limiting value 0 at 0° and 180°, and its limiting absolute value 1 at 90° and 270°.

If we have an equation in which the value of the sine is found to be greater than 1 or less than -1, we know either that the equation is wrong or that an error has been made in the solution.

Of course the values of the functions of 360° are the same as those of 0°, since the moving radius has returned to its original position and the initial and terminal sides of the angle coincide.

In the same way, the absolute value of the cosine cannot be greater than 1, and it has its limiting value 0 at 90° and 270°, and its limiting absolute value 1 at 0° and 180°. Similarly we can find the limiting values of all the other functions.

For convenience we speak of ∞ as a limiting value, although the function increases without limit, the meaning of the expression in this case being clear.

Summarizing these results, we have the following table:

Function	0°	90°	180°	270°	360°
Sine	∓ 0	$+1$	± 0	-1	∓ 0
Cosine	$+1$	± 0	-1	∓ 0	$+1$
Tangent	∓ 0	$\pm \infty$	∓ 0	$\pm \infty$	∓ 0
Cotangent	$\mp \infty$	± 0	$\mp \infty$	± 0	$\mp \infty$
Secant	$+1$	$\pm \infty$	-1	$\mp \infty$	$+1$
Cosecant	$\mp \infty$	$+1$	$\pm \infty$	-1	$\mp \infty$

Sines and cosines vary in value from $+1$ to -1; tangents and cotangents, from $+\infty$ to $-\infty$; secants and cosecants, from $+\infty$ to $+1$, and from -1 to $-\infty$.

In the table given above the double sign $\pm$ or $\mp$ is placed before 0 and ∞. From the preceding investigation it appears that the functions *always change sign in passing through 0 or through* ∞; and the sign $\pm$ or $\mp$ prefixed to 0 or ∞ simply shows the direction from which the value is reached. For example, at 0° the sine is passing from $-$ (in quadrant IV) to $+$ (in quadrant I). At 90° the tangent is passing from $+$ (in quadrant I) to $-$ (in quadrant II).

85. Functions of Angles Greater than 360°. The functions of $360° + x$ are the same in sign and in absolute value as those of x. If n is a positive integer,

The functions of $(n \times 360° + x)$ *are the same as those of* x.

For example, the functions of 2200°, or $6 \times 360° + 40°$, are the same in sign and in absolute value as the functions of 40°.

Exercise 37. Variations in the Functions

Represent the following functions by lines in a unit circle:

1. $\sin 135°$.	7. $\sin 210°$.	13. $\sin 300°$.	19. $\sin 270°$.
2. $\cos 120°$.	8. $\cos 225°$.	14. $\cos 315°$.	20. $\cos 180°$.
3. $\tan 150°$.	9. $\tan 240°$.	15. $\tan 330°$.	21. $\tan 180°$.
4. $\cot 135°$.	10. $\cot 210°$.	16. $\cot 300°$.	22. $\cot 270°$.
5. $\sec 120°$.	11. $\sec 225°$.	17. $\sec 315°$.	23. $\sec 180°$.
6. $\csc 150°$.	12. $\csc 240°$.	18. $\csc 330°$.	24. $\csc 270°$.

25. Prepare a table showing the signs of all the functions in each of the four quadrants.

26. Prepare a table showing which functions always have the minus sign in each of the four quadrants.

Represent the following functions by lines in a unit circle:

27. $\sin 390°$.	30. $\cos 390°$.	33. $\sin 460°$.	36. $\tan 475°$.
28. $\tan 405°$.	31. $\cot 405°$.	34. $\sin 570°$.	37. $\sec 705°$.
29. $\sec 420°$.	32. $\csc 420°$.	35. $\sin 720°$.	38. $\csc 810°$.

Show by lines in a unit circle that:

39. $\sin 150° = \sin 30°$.	45. $\tan 120° = -\tan 60°$.
40. $\cos 150° = -\cos 30°$.	46. $\cot 120° = -\cot 60°$.
41. $\sin 210° = -\sin 30°$.	47. $\tan 240° = \tan 60°$.
42. $\cos 210° = -\cos 30°$.	48. $\cot 240° = \cot 60°$.
43. $\sin 330° = -\sin 30°$.	49. $\tan 300° = -\tan 60°$.
44. $\cos 330° = \cos 30°$.	50. $\cot 300° = -\cot 60°$.

51. Write the signs of the functions of the following angles: 340°, 239°, 145°, 400°, 700°, 1200°, 3800°.

52. How many values less than 360° can the angle x have if $\sin x = +\frac{5}{7}$, and in what quadrants do the angles lie? Draw a figure.

53. How many values less than 720° can the angle x have if $\cos x = +\frac{2}{3}$, and in what quadrants do the angles lie? Draw a figure.

54. If we take into account only angles less than 180°, how many values can x have if $\sin x = \frac{5}{7}$? if $\cos x = \frac{1}{5}$? if $\cos x = -\frac{4}{5}$? if $\tan x = \frac{2}{3}$? if $\cot x = -7$?

55. Within what limits between 0° and 360° must the angle x lie if $\cos x = -\frac{2}{9}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$?

56. Why may $\cot 360°$ be considered as either $+\infty$ or $-\infty$?

57. Find the values of $\sin 450°$, $\tan 540°$, $\cos 630°$, $\cot 720°$, $\sin 810°$, $\csc 900°$, $\cos 1800°$, $\sin 3600°$.

58. What functions of an angle of a triangle may be negative? In what cases are they negative?

59. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative?

60. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{3}{5}$? Of these sines how many are positive?

Compute the values of the following expressions:

61. $a \sin 0° + b \cos 90° - c \tan 180°$.

62. $a \cos 90° - b \tan 180° + c \cot 90°$.

63. $a \sin 90° - b \cos 360° + (a - b) \cos 180°$.

64. $(a^2 - b^2) \cos 360° - 4\,ab \sin 270° + \sin 360°$.

65. $(a^2 + b^2) \cos 180° + (a^2 + b^2) \sin 180° + (a^2 + b^2) \tan 135°$.

66. $(a^2 + 2\,ab + b^2) \sin 90° + (a^2 - 2\,ab + b^2) \cos 180° - 4\,ab \tan 225°$.

67. $(a - b + c - d) \sin 270° - (a - b + c - d) \cos 180° + a \tan 360°$.

State the sign of each of the six functions of the following angles:

68. 75°. **70.** 155°. **72.** 275°. **74.** 355°.

69. 125°. **71.** 185°. **73.** 325°. **75.** $-65°$.

Find the four smallest angles that satisfy the following conditions:

76. $\sin A = \frac{1}{2}$. **78.** $\sin A = \frac{1}{2}\sqrt{3}$. **80.** $\tan A = \frac{1}{3}\sqrt{3}$.

77. $\cos A = \frac{1}{2}\sqrt{3}$. **79.** $\cos A = \frac{1}{2}$. **81.** $\tan A = \sqrt{3}$.

Find two angles less than 360° that satisfy the following conditions:

82. $\sin A = -\frac{1}{2}$. **84.** $\sin A = -\frac{1}{2}\sqrt{2}$. **86.** $\tan A = -1$.

83. $\cos A = -\frac{1}{2}$. **85.** $\cos A = -\frac{1}{2}\sqrt{2}$. **87.** $\cot A = -1$.

If A, B, and C are the angles of any triangle ABC, prove that:

88. $\cos \frac{1}{2}A = \sin \frac{1}{2}(B + C)$. **90.** $\cos \frac{1}{2}B = \sin \frac{1}{2}(A + C)$.

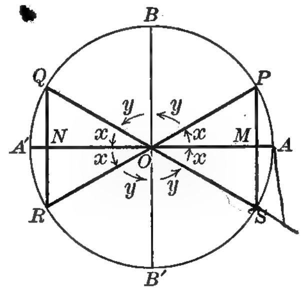
89. $\sin \frac{1}{2}C = \cos \frac{1}{2}(A + B)$. **91.** $\sin \frac{1}{2}A = \cos \frac{1}{2}(B + C)$.

As angle A increases from 0° to 360°, trace the changes in sign and magnitude of the following:

92. $\sin A \cos A$. **94.** $\sin A - \cos A$. **96.** $\tan A + \cot A$.

93. $\sin A + \cos A$. **95.** $\sin A \div \cos A$. **97.** $\tan A - \cot A$.

86. Reduction of Functions to the First Quadrant. In the annexed figure BB' is perpendicular to the horizontal diameter AA', and the diameters PR and QS are so drawn as to make $\angle AOP = \angle SOA$. It therefore follows from geometry that △ MOP, MOS, NOQ, and NOR are congruent.



Considering, therefore, only the *absolute values* of the functions, we have

$\sin AOP = \sin AOQ = \sin AOR = \sin AOS$,
$\cos AOP = \cos AOQ = \cos AOR = \cos AOS$,

and so on for the other functions.

Hence, *For every acute angle there is an angle in each of the higher quadrants whose functions, in absolute value, are equal to those of this acute angle.*

If we let $\angle AOP = x$ and $\angle POB = y$, noticing that $\angle AOP = \angle QOA' = \angle A'OR = \angle SOA = x$, and $\angle POB = \angle BOQ = \angle ROB' = \angle B'OS = y$, and prefixing the proper signs to the functions (§ 83), we have:

Angle in Quadrant II

$\sin(180° - x) = \sin x$	$\sin(90° + y) = \cos y$
$\cos(180° - x) = -\cos x$	$\cos(90° + y) = -\sin y$
$\tan(180° - x) = -\tan x$	$\tan(90° + y) = -\cot y$
$\cot(180° - x) = -\cot x$	$\cot(90° + y) = -\tan y$

Angle in Quadrant III

$\sin(180° + x) = -\sin x$	$\sin(270° - y) = -\cos y$
$\cos(180° + x) = -\cos x$	$\cos(270° - y) = -\sin y$
$\tan(180° + x) = \tan x$	$\tan(270° - y) = \cot y$
$\cot(180° + x) = \cot x$	$\cot(270° - y) = \tan y$

Angle in Quadrant IV

$\sin(360° - x) = -\sin x$	$\sin(270° + y) = -\cos y$
$\cos(360° - x) = \cos x$	$\cos(270° + y) = \sin y$
$\tan(360° - x) = -\tan x$	$\tan(270° + y) = -\cot y$
$\cot(360° - x) = -\cot x$	$\cot(270° + y) = -\tan y$

For example, $\sin 127° = \sin(180° - 53°) = \sin 53° = \cos 37°$,
$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\cos 60°$,
and $\sin 350° = \sin(360° - 10°) = -\sin 10° = -\cos 80°$.

It appears from the results set forth on page 90 that *the functions of any angle, however great, can be reduced to the functions of an angle in the first quadrant.*

For example, suppose that we have a polygon with a reëntrant angle of 247° 30′, and we wish to find the tangent of this angle. We may proceed by finding $\tan(180° + x)$ or by finding $\tan(270° - x)$. We then have

$$\tan 247° 30' = \tan(180° + 67° 30') = \tan 67° 30',$$

and

$$\tan 247° 30' = \tan(270° - 22° 30') = \cot 22° 30'.$$

That these two results are equal is apparent, for

$$\tan 67° 30' = \cot(90° - 67° 30') = \cot 22° 30'.$$

It also appears that, *for angles less than 180°, a given value of a sine or cosecant determines two supplementary angles, one acute, the other obtuse; a given value of any other function determines only one angle, this angle being acute if the value is positive and obtuse if the value is negative.*

For example, if we know that $\sin x = \frac{1}{2}$, we cannot tell whether $x = 30°$ or 150°, since the sine of each of these angles is $\frac{1}{2}$. But if we know that $\tan x = 1$, we know that $x = 45°$.

Similarly, if we know that $\cot x = -1$, we know that $x = 135°$, there being no other angle less than 180° whose cotangent is -1.

Since $\sec x$ is the reciprocal of $\cos x$ and $\csc x$ is the reciprocal of $\sin x$, and since by the aid of logarithms we can divide by $\cos x$ or $\sin x$ as easily as we can multiply by $\sec x$ or $\csc x$, we shall hereafter pay but little attention to the secant and cosecant. Since the invention of logarithms these functions have been of little practical importance in the work of ordinary mensuration.

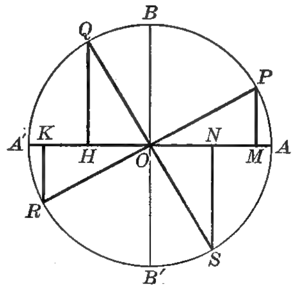
Exercise 38. Reduction to the First Quadrant

Express the following as functions of angles less than 90°:

1. $\sin 170°$.	11. $\sin 275°$.	21. $\sin 148° 10'$.
2. $\cos 160°$.	12. $\sin 345°$.	22. $\cos 192° 20'$.
3. $\tan 148°$.	13. $\tan 282°$.	23. $\tan 265° 30'$.
4. $\cot 156°$.	14. $\tan 325°$.	24. $\cot 287° 40'$.
5. $\sin 180°$.	15. $\cos 290°$.	25. $\sin 187° 10' 3''$.
6. $\tan 180°$.	16. $\cos 350°$.	26. $\cos 274° 5' 14''$.
7. $\sin 200°$.	17. $\cot 295°$.	27. $\tan 322° 8' 15''$.
8. $\cos 225°$.	18. $\cot 347°$.	28. $\cot 375° 10' 3''$.
9. $\tan 258°$.	19. $\sin 360°$.	29. $\sin 147.75°$.
10. $\cot 262°$.	20. $\cos 360°$.	30. $\cos 232.25°$.

87. Functions of Angles Differing by 90°. It was shown in the case of acute angles that the function of any angle is equal to the co-function of its complement (§ 8).

That is, $\tan 28° = \cot(90° - 28°) = \cot 62°$,
$\sin x = \cos(90° - x)$, and so on.



It will now be shown for all angles that *if two angles differ by 90°, the functions of either are equal in absolute value to the co-functions of the other.*

In the annexed figure the diameters PR and QS are perpendicular to each other, and from P, Q, R, and S perpendiculars are drawn to AA'. Then from the congruent triangles OMP, QHO, OKR, and SNO we see that

$$OM = QH = OK = SN,$$

and

$$MP = OH = KR = ON.$$

Hence, considering the proper signs (§ 83),

$$\sin AOQ = \cos AOP, \qquad \cos AOQ = -\sin AOP,$$
$$\sin AOR = \cos AOQ, \qquad \cos AOR = -\sin AOQ,$$
$$\sin AOS = \cos AOR, \qquad \cos AOS = -\sin AOR.$$

In all these equations, if x denotes the angle on the right-hand side, the angle on the left-hand side is $90° + x$.

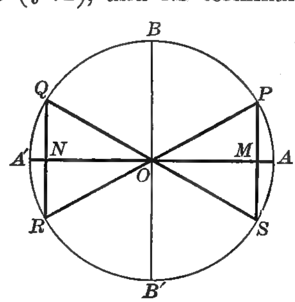
Therefore, if x is an angle in any one of the four quadrants,

$$\sin(90° + x) = \cos x, \qquad \cos(90° + x) = -\sin x;$$

and hence $\tan(90° + x) = -\cot x, \qquad \cot(90° + x) = -\tan x.$

It is therefore seen that the algebraic sign of the function of the resulting angle is the same as that found in the similar case in § 86.

88. Functions of a Negative Angle. If the angle x is generated by the radius moving clockwise from the initial position OA to the terminal position OS, it will be negative (§ 72), and its terminal side will be identical with that for the angle $360° - x$. Therefore the functions of the angle $-x$ are the same as those of the angle $360° - x$; or



$$\sin(-x) = -\sin x,$$
$$\cos(-x) = \cos x,$$
$$\tan(-x) = -\tan x,$$
$$\cot(-x) = -\cot x.$$

Exercise 39. Reduction of Functions

Express the following as functions of angles less than 45°:

1. $\sin 100°$.
2. $\sin 120°$.
3. $\sin 110°$.
4. $\sin 130°$.
5. $\cos 95°$.
6. $\cos 97°$.
7. $\cos 111°$.
8. $\cos 127°$.
9. $\tan 91°$.
10. $\tan 99°$.
11. $\tan 119°$.
12. $\tan 129°$.
13. $\cot 94°\ 1'$.
14. $\cot 97°\ 2'$.
15. $\cot 98°\ 3'$.
16. $\cot 99°\ 9'$.

Express the following as functions of positive angles:

17. $\sin(-3°)$.
18. $\sin(-9°)$.
19. $\sin(-86°)$.
20. $\cos(-75°)$.
21. $\cos(-87°)$.
22. $\cos(-95°)$.
23. $\tan(-100°)$.
24. $\tan(-150°)$.
25. $\tan(-200°)$.
26. $\cot(-1.5°)$.
27. $\cot(-7.8°)$.
28. $\cot(-9.1°)$.

Find the following by aid of the tables:

29. $\sin 178°\ 30'$.
30. $\cos 236°\ 45'$.
31. $\tan 322°\ 18'$.
32. $\cot 423°\ 15'$.
33. $\sin(-7°\ 29'\ 30'')$.
34. $\cos(-29°\ 42'\ 19'')$.
35. $\tan(-172°\ 16'\ 14'')$.
36. $\cot(-262°\ 17'\ 15'')$.
37. $\log \sin 127.5°$.
38. $\log \cos 226.4°$.
39. $\log \tan 327.8°$.
40. $\log \cot 343.3°$.
41. $\log \sin 236°\ 13'\ 5''$.
42. $\log \cos 327°\ 5'\ 11''$.
43. $\log \tan(-125°\ 27')$.
44. $\log \cot(-236°\ 15')$.

45. Show that the angles 42°, 138°, −318°, 402°, and −222° all have the same sine.

46. Find four angles between 0° and 720° which satisfy the equation $\sin x = -\frac{1}{2}\sqrt{2}$.

47. Draw a circle with unit radius, and represent by lines the sine, cosine, tangent, and cotangent of −325°.

48. Show by drawing a figure that $\sin 195° = \cos(-105°)$, and that $\cos 300° = \sin(-210°)$.

49. Show by drawing a figure that $\cos 320° = -\cos(-140°)$, and that $\sin 320° = -\sin 40°$.

50. Show by drawing a figure that $\sin 765° = \frac{1}{2}\sqrt{2}$, and that $\tan 1395° = -1$.

51. In the triangle ABC show that $\cos A = -\cos(B + C)$, and that $\cos B = -\cos(A + C)$.

89. Relations of the Functions. Certain relations between the functions have already been proved to exist in the case of acute angles (§§ 13, 14), and since the relations of the functions of any angle to the functions of an acute angle have also been considered (§§ 80, 85, 86, 88), it is evident that the laws are true for any angle. These laws are so important that they will now be summarized, and others of a similar kind will be added.

These laws should be memorized. They will be needed frequently in the subsequent work. The proof of each should be given, as required in § 14. The $\pm$ sign is placed before the square root sign, since we have now learned the meaning of negative functions.

To find the sine we have:

$$\sin x = \frac{1}{\csc x} \qquad \sin x = \pm\sqrt{1-\cos^2 x}$$

To find the cosine we have:

$$\cos x = \frac{1}{\sec x} \qquad \cos x = \pm\sqrt{1-\sin^2 x}$$

To find the tangent we have:

$$\tan x = \frac{1}{\cot x} \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\tan x = \pm\frac{\sin x}{\sqrt{1-\sin^2 x}} \qquad \tan x = \pm\frac{\sqrt{1-\cos^2 x}}{\cos x}$$

$$\tan x = \pm\sqrt{\sec^2 x - 1} \qquad \tan x = \sin x \sec x$$

To find the cotangent we have:

$$\cot x = \frac{1}{\tan x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\cot x = \pm\frac{\cos x}{\sqrt{1-\cos^2 x}} \qquad \cot x = \pm\frac{\sqrt{1-\sin^2 x}}{\sin x}$$

$$\cot x = \pm\sqrt{\csc^2 x - 1} \qquad \cot x = \cos x \csc x$$

To find the secant we have:

$$\sec x = \frac{1}{\cos x} \qquad \sec x = \pm\sqrt{1+\tan^2 x}$$

To find the cosecant we have:

$$\csc x = \frac{1}{\sin x} \qquad \csc x = \pm\sqrt{1+\cot^2 x}$$

Exercise 40. Relations of the Functions

1. Prove each of the formulas given in § 89.

Prove the following relations:

2. $\sin x = \pm \dfrac{\tan x}{\sqrt{1+\tan^2 x}}$.

3. $\cos x = \pm \dfrac{\cot x}{\sqrt{1+\cot^2 x}}$.

4. $\tan x = \pm \dfrac{1}{\sqrt{\csc^2 x - 1}}$.

5. $\cot x = \pm \dfrac{1}{\sqrt{\sec^2 x - 1}}$.

6. Find $\sin x$ in terms of $\cot x$.

7. Find $\cos x$ in terms of $\tan x$.

8. Find $\sec x$ in terms of $\sin x$.

9. Find $\csc x$ in terms of $\cos x$.

Prove the following relations:

10. $\tan x \cos x = \sin x$.

11. $\cos^2 x = \cot^2 x - \cot^2 x \cos^2 x$.

12. $\tan^2 x = \sin^2 x + \sin^2 x \tan^2 x$.

13. $\cos^2 x + 2\sin^2 x = 1 + \sin^2 x$.

14. $\cot^2 x = \cos^2 x + \cos^2 x \cot^2 x$.

15. $\cot^2 x \sec^2 x = 1 + \cot^2 x$.

16. $\csc^2 x - \cot^2 x = 1$.

17. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$.

18. Show that the sum of the tangent and cotangent of an angle is equal to the product of the secant and cosecant of the angle.

Recalling the values given on page 8, find the value of x when:

19. $2\cos x = \sec x$.

20. $4\sin x = \csc x$.

21. $\sin^2 x = 3\cos^2 x$.

22. $2\sin^2 x + \cos^2 x = \frac{3}{2}$.

23. $3\tan^2 x - \sec^2 x = 1$.

24. $\tan x + \cot x = 2$.

25. $\tan x = 2\sin x$.

26. $\sec x = \sqrt{2}\tan x$.

27. $\sin^2 x - \cos x = \frac{1}{4}$.

28. $\tan^2 x - \sec x = 1$.

29. $\tan^2 x + \csc^2 x = 3$.

30. $\sin x + \sqrt{3}\cos x = 2$.

31. Given $(\sin x + \cos x)^2 - 1 = (\sin x - \cos x)^2 + 1$, find x.

32. Given $2\sin x = \cos x$, find $\sin x$ and $\cos x$.

33. Given $4\sin x = \tan x$, find $\sin x$ and $\tan x$.

34. Given $5\sin x = \tan x$, find $\cos x$ and $\sec x$.

35. Given $4\cot x = \tan x$, find the other functions.

36. Given $\sin x = 4\cos x$, find $\sin x$ and $\cos x$.

37. If $\sin x : \cos x = 9 : 40$, find $\sin x$ and $\cos x$.

38. From the formula $\tan x = \pm \dfrac{\sin x}{\sqrt{1-\sin^2 x}}$, find the condition under which $\tan x = \sin x$.

Solve the following equations; that is, find the value of x when:

39. $\cos x = \sec x$.
40. $\cos x = \tan x$.
41. $\cos x = \sin x$.
42. $\tan x = \cot x$.
43. $\sec x = \csc x$.
44. $2 \cos x + \sec x = 3$.
45. $\cos^2 x - \sin^2 x = \sin x$.
46. $2 \sin x + \cot x = 1 + 2 \cos x$.
47. $\sin^2 x + \tan^2 x = 3 \cos^2 x$.
48. $\tan x + 2 \cot x = \frac{5}{2} \csc x$.

Prove the following relations:

49. $\sin A + \cos A = (1 + \tan A) \cos A$.
50. $\dfrac{\cot x}{\cos x} = \sqrt{1 + \cot^2 x}$.
51. $\cos x : \cot x = \sqrt{1 - \cos^2 x}$.
52. $\tan^2 x = \dfrac{1}{\cos^2 x} - 1$.

Find the values of the other functions of A when:

53. $\sin A = \frac{2}{3}$.
54. $\cos A = \frac{3}{4}$.
55. $\tan A = 1.5$.
56. $\cot A = 0.75$.
57. $\sec A = 1.5$.
58. $\sin A = \frac{12}{13}$.
59. $\sin A = 0.8$.
60. $\cos A = \frac{60}{61}$.
61. $\cos A = 0.28$.
62. $\tan A = \frac{4}{3}$.
63. $\cot A = 1$.
64. $\cot A = 0.5$.
65. $\sec A = 2$.
66. $\csc A = \sqrt{2}$.
67. $\sin A = m$.
68. Given $\sin A = 2m : (1 + m^2)$, find the value of $\tan A$.
69. Given $\cos A = 2mn : (m^2 + n^2)$, find the value of $\sec A$.
70. Given $\sin 0° = 0$, find the other functions of 0°.
71. Given $\sin 90° = 1$, find the other functions of 90°.
72. Given $\tan 90° = \infty$, find the other functions of 90°.
73. Given $\cot 22° 30' = \sqrt{2} + 1$, find the other functions of 22° 30′.
74. Write $\tan^2 A + \cot^2 A$ so as to contain only $\cos A$.

In the triangle ABC, prove the following relations:

75. $\sin A = \sin (B + C)$.
76. $\cos A = - \cos (B + C)$.
77. $\tan A = - \tan (B + C)$.
78. $\cot A = - \cot (B + C)$.
79. $\sin A = - \sin (2A + B + C)$.
80. $\sin B = - \sin (A + 2B + C)$.
81. $\cos C = - \cos (A + B + 2C)$.
82. $\cot B = \cot (A + 2B + C)$.
83. $\sin A = - \cos (\frac{3}{2}A + \frac{1}{2}B + \frac{1}{2}C)$.
84. $\cos A = - \cos (2A + B + C)$.
85. $\cos A = \sin (\frac{3}{2}A + \frac{1}{2}B + \frac{1}{2}C)$.
86. $\sin (\frac{1}{2}A + B) = \cos (\frac{1}{2}B - \frac{1}{2}C)$.
87. $\sin (\frac{1}{2}C - \frac{1}{2}A) = - \cos (\frac{1}{2}B + C)$
88. $\cos B = - \cos (A + 2B + C)$.
89. $\tan A = \tan (2A + B + C)$.
90. $\cot A = \tan (\frac{3}{2}B + \frac{3}{2}C + \frac{1}{2}A)$.

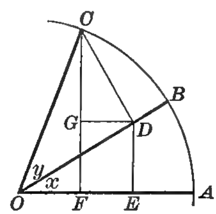
In the quadrilateral $ABCD$, prove the following relations:

91. $-\sin A = \sin (B + C + D)$.
92. $\cos A = \cos (B + C + D)$.
93. $-\tan A = \tan (B + C + D)$.
94. $-\cot A = \cot (B + C + D)$.

CHAPTER VI

FUNCTIONS OF THE SUM OR THE DIFFERENCE OF TWO ANGLES

90. Formula for $\sin(x+y)$. In this figure there are shown two acute angles, x and y, with $\angle AOC$ acute and equal to $x+y$; two perpendiculars are let fall from C, and two from D, as shown. Then by geometry the triangles CGD and EOD are similar and hence $\angle GCD = \angle EOD = x$. Considering the radius as unity, $OD = \cos y$ and $CD = \sin y$. Hence we have



$$\sin(x+y) = CF = DE + CG.$$

But $\sin x = \dfrac{DE}{OD}$, whence $DE = \sin x \cdot OD$
$= \sin x \cos y$;

and $\cos x = \dfrac{CG}{CD}$, whence $CG = \cos x \cdot CD$
$= \cos x \sin y$.

Hence $$\mathbf{\sin(x+y) = \sin x \cos y + \cos x \sin y.}$$

This is one of the most important formulas and should be memorized.

For example, $\sin(30° + 60°) = \sin 30° \cos 60° + \cos 30° \sin 60°$

$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = 1,$$

which we have already found to be $\sin 90°$.

91. Formula for $\cos(x+y)$. Using the above figure we see that

$$\cos(x+y) = OF = OE - DG.$$

But $\cos x = \dfrac{OE}{OD}$, whence $OE = \cos x \cdot OD = \cos x \cos y$;

and $\sin x = \dfrac{DG}{CD}$, whence $DG = \sin x \cdot CD = \sin x \sin y$.

Hence $$\mathbf{\cos(x+y) = \cos x \cos y - \sin x \sin y.}$$

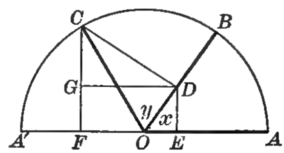
This important formula should be memorized.

For example, $\cos(45° + 45°) = \cos 45° \cos 45° - \sin 45° \sin 45°$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0,$$

which we have already found to be $\cos 90°$.

92. The Proofs continued. In the proofs given on page 97, x, y, and $x+y$ were assumed to be acute angles. If, however, x and y are acute but $x+y$ is obtuse, as shown in this figure, the proofs remain, word for word, the same as before, the only difference being that the sign of OF will be negative, as DG is now greater than OE. This, however, does not affect the proof. The above formulas, therefore, hold true for all acute angles x and y.



Furthermore, if these formulas hold true for any two acute angles x and y, they hold true when one of the angles is increased by 90°. Thus, if for x we write $x' = 90° + x$, then, by § 87,

$$\sin(x'+y) = \sin(90° + x + y) = \cos(x+y)$$
$$= \cos x \cos y - \sin x \sin y.$$

But by § 87, $\quad \cos x = \sin(90° + x) = \sin x',$

and $\quad \sin x = -\cos(90° + x) = -\cos x'.$

Hence, by substituting these values,

$$\sin(x'+y) = \sin x' \cos y + \cos x' \sin y.$$

That is, § 90 holds true if either angle is repeatedly increased by 90°. It is therefore true for all angles.

Similarly, by § 87,

$$\cos(x'+y) = \cos(90° + x + y) = -\sin(x+y)$$
$$= -\sin x \cos y - \cos x \sin y$$
$$= \cos x' \cos y - \sin x' \sin y,$$

by substituting $\cos x'$ for $-\sin x$ and $\sin x'$ for $\cos x$ as above.

That is, § 91 also holds true if either angle is repeatedly increased by 90°. It is therefore true for all angles.

Exercise 41. Sines and Cosines

Given $\sin 30° = \cos 60° = \frac{1}{2}$, $\cos 30° = \sin 60° = \frac{1}{2}\sqrt{3}$, *and* $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}$, *find the values of the following:*

1. $\sin 15°$.	5. $\sin 90°$.	9. $\sin 120°$.	13. $\sin 150°$.
2. $\cos 15°$.	6. $\cos 90°$.	10. $\cos 120°$.	14. $\cos 150°$.
3. $\sin 75°$.	7. $\sin 105°$.	11. $\sin 135°$.	15. $\sin 165°$.
4. $\cos 75°$.	8. $\cos 105°$.	12. $\cos 135°$.	16. $\cos 165°$.

93. Formula for $\tan(x+y)$. Since $\tan A = \dfrac{\sin A}{\cos A}$, therefore

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y},$$

whatever the size of the angles x and y (§ 92).

Dividing each term of the numerator and denominator of the last of these fractions by $\cos x \cos y$, we have

$$\tan(x+y) = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}}.$$

But since $\qquad \dfrac{\sin x}{\cos x} = \tan x$, and $\dfrac{\sin y}{\cos y} = \tan y$,

we have $\qquad \boldsymbol{\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}}.$

This important formula should be memorized.

94. Formula for $\cot(x+y)$. Since $\cot A = \dfrac{\cos A}{\sin A}$, therefore

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y},$$

whatever the size of the angles x and y (§ 92).

Dividing each term of the numerator and denominator of the last of these fractions by $\sin x \sin y$, and then remembering that $\dfrac{\cos x}{\sin x} = \cot x$ and $\dfrac{\cos y}{\sin y} = \cot y$, we have

$$\boldsymbol{\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}}.$$

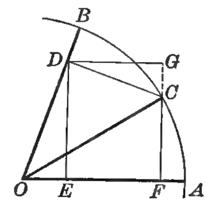
This important formula should be memorized.

Exercise 42. Tangents and Cotangents

Given $\tan 30° = \cot 60° = \frac{1}{3}\sqrt{3}$, $\cot 30° = \tan 60° = \sqrt{3}$, $\tan 45° = \cot 45° = 1$, find the values of the following:

1. $\tan 15°$.	5. $\tan 90°$.	9. $\tan 120°$.	13. $\tan 150°$.
2. $\cot 15°$.	6. $\cot 90°$.	10. $\cot 120°$.	14. $\cot 150°$.
3. $\tan 75°$.	7. $\tan 105°$.	11. $\tan 135°$.	15. $\tan 165°$.
4. $\cot 75°$.	8. $\cot 105°$.	12. $\cot 135°$.	16. $\cot 165°$

95. Formula for sin $(x-y)$. In this figure there are shown two acute angles, $AOB = x$ and $COB = y$, with $\angle AOC$ equal to $x - y$; two perpendiculars are let fall from C, and two from D.



The perpendiculars from D are DE and DG, DG being drawn to FC produced.

Then, considering the radius as unity, we have

$$\sin(x-y) = CF = DE - CG.$$

But $$DE = \sin x \cdot OD = \sin x \cos y,$$

and $$GC = \cos x \cdot CD = \cos x \sin y.$$

Hence, by substituting these values of DE and GC,

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

This is one of the most important formulas and should be memorized.

96. Formula for cos $(x-y)$. Using the above figure we see that

$$\cos(x-y) = OF = OE + DG.$$

But $$OE = \cos x \cdot OD = \cos x \cos y,$$

and $$DG = \sin x \cdot CD = \sin x \sin y.$$

Hence it follows that

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

This important formula should be memorized. The proof in §§ 95 and 96 refers only to acute angles, but the formulas are entirely general if due regard is paid to the algebraic signs. The general proof may follow the method of § 92, or it may be based upon it; the latter plan is followed in § 97.

97. The Proofs continued. Since $x = (x-y) + y$, we see that

$$\sin x = \sin\{(x-y)+y\} = \sin(x-y)\cos y + \cos(x-y)\sin y,$$
$$\cos x = \cos\{(x-y)+y\} = \cos(x-y)\cos y - \sin(x-y)\sin y.$$

Multiplying the first equation by $\cos y$, and the second by $\sin y$,

$$\sin x \cos y = \sin(x-y)\cos^2 y + \cos(x-y)\sin y \cos y,$$
$$\cos x \sin y = -\sin(x-y)\sin^2 y + \cos(x-y)\sin y \cos y.$$

Hence $\quad \sin x \cos y - \cos x \sin y = \sin(x-y)(\sin^2 y + \cos^2 y).$

But by § 14 $\qquad \sin^2 y + \cos^2 y = 1.$

Therefore $\qquad \sin(x-y) = \sin x \cos y - \cos x \sin y.$

Similarly, $\qquad \cos(x-y) = \cos x \cos y + \sin x \sin y.$

Therefore the formulas of §§ 95 and 96 are universally true.

98. Formula for tan $(x - y)$. Since $\tan A = \dfrac{\sin A}{\cos A}$, we have

$$\tan(x-y) = \frac{\sin(x-y)}{\cos(x-y)}$$
$$= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}.$$

Dividing numerator and denominator by $\cos x \cos y$, as in § 93, we obtain

$$\tan(x-y) = \frac{\dfrac{\sin x}{\cos x} - \dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin y}{\cos x} \cdot \dfrac{\sin y}{\cos y}}.$$

That is,
$$\boldsymbol{\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}}.$$

This important formula should be memorized.

99. Formula for cot $(x - y)$. Following the plan suggested in § 98, we may show that

$$\cot(x-y) = \frac{\cos(x-y)}{\sin(x-y)}$$
$$= \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y - \cos x \sin y}$$
$$= \frac{\dfrac{\cos x}{\sin x} \cdot \dfrac{\cos y}{\sin y} + 1}{\dfrac{\cos y}{\sin y} - \dfrac{\cos x}{\sin x}}.$$

That is,
$$\boldsymbol{\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}}.$$

This important formula should be memorized.

100. Summary of the Addition Formulas. The formulas of §§ 90–99 may be combined as follows:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}.$$

When the signs $\pm$ and $\mp$ occur in the same formula we should be careful to take the $-$ of $\mp$ with the $+$ of $\pm$. That is, the upper signs are to be taken together, and the lower signs are to be taken together.

Exercise 43. The Addition Formulas

Given $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{5}{13}$, $\cos y = \frac{12}{13}$, *find the value of:*

1. $\sin(x + y)$.
2. $\sin(x - y)$.
3. $\cos(x + y)$.
4. $\cos(x - y)$.
5. $\tan(x + y)$.
6. $\tan(x - y)$.

By letting $x = 90°$ *in the formulas, find the following:*

7. $\sin(90° - y)$.
8. $\cos(90° - y)$.
9. $\tan(90° - y)$.

Similarly, by substituting in the formulas, find the following:

10. $\sin(90° + y)$.
11. $\sin(180° - y)$.
12. $\sin(180° + y)$.
13. $\sin(270° - y)$.
14. $\sin(270° + y)$.
15. $\sin(360° - y)$.
16. $\sin(360° + y)$.
17. $\cos(x - 90°)$.
18. $\cos(x - 180°)$.
19. $\cos(x - 270°)$.
20. $\tan(x - 90°)$.
21. $\tan(x - 180°)$.
22. $\cot(x - 90°)$.
23. $\cot(x - 180°)$.
24. $\sin(-y)$.
25. $\sin(45° - y)$.
26. $\cos(45° - y)$.
27. $\tan(45° - y)$.
28. $\cot(30° + y)$.
29. $\cot(60° - y)$.
30. $\cot(90° - y)$.

31. If $\tan x = 0.5$ and $\tan y = 0.25$, find $\tan(x + y)$ and $\tan(x - y)$.

32. If $\tan x = 1$ and $\tan y = \frac{1}{3}\sqrt{3}$, find $\tan(x + y)$ and $\tan(x - y)$.

33. If $\tan x = \frac{5}{6}$ and $\tan y = \frac{1}{11}$, find $\tan(x + y)$ and $\tan(x - y)$, and find the number of degrees in $x + y$.

34. If $\tan x = 2$ and $\tan y = \frac{1}{2}$, what is the nature of the angle $x + y$? Consider the same question when $\tan x = 3$ and $\tan y = \frac{1}{3}$, and when $\tan x = a$ and $\tan y = 1/a$.

35. Prove that the sum of $\tan(x - 45°)$ and $\cot(x + 45°)$ is zero.

36. Prove that the sum of $\cot(x - 45°)$ and $\tan(x + 45°)$ is zero.

37. If $\sin x = 0.2\sqrt{5}$ and $\sin y = 0.1\sqrt{10}$, prove that $x + y = 45°$. May $x + y$ have other values? If so, state two of these values.

38. Prove that if an angle x is decreased by 45° the cotangent of the resulting angle is equal to $-\dfrac{\cot x + 1}{\cot x - 1}$.

39. Prove that if an angle x is increased by 45° the cotangent of the resulting angle is equal to $\dfrac{\cot x - 1}{\cot x + 1}$.

40. If $\tan x = \dfrac{a}{1 + a}$ and $\tan y = \dfrac{1}{1 + 2a}$, prove that $\tan(x + y) = 1$.

41. If a right angle is divided into any three angles x, y, z, prove that $\tan x = \dfrac{1 - \tan y \tan z}{\tan y + \tan z}$.

101. Functions of Twice an Angle. By substituting in the formulas for the functions of $x + y$ we obtain the following important formulas for the functions of twice an angle:

$$\sin 2x = 2 \sin x \cos x,$$

$$\cos 2x = \cos^2 x - \sin^2 x,$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x},$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.$$

Letting $2x = y$ we have the following useful formulas:

$$\sin y = 2 \sin \tfrac{1}{2} y \cos \tfrac{1}{2} y, \qquad \cos y = \cos^2 \tfrac{1}{2} y - \sin^2 \tfrac{1}{2} y,$$

$$\tan y = \frac{2 \tan \frac{1}{2} y}{1 - \tan^2 \frac{1}{2} y}, \qquad \cot y = \frac{\cot^2 \frac{1}{2} y - 1}{2 \cot \frac{1}{2} y}.$$

Exercise 44. Functions of Twice an Angle

As suggested above, deduce the formulas for the following:

1. $\sin 2x$. 2. $\cos 2x$. 3. $\tan 2x$. 4. $\cot 2x$.

Find $\sin 2x$, given the following values of $\sin x$ and $\cos x$:

5. $\sin x = \frac{1}{2}\sqrt{2}$, $\cos x = \frac{1}{2}\sqrt{2}$. 6. $\sin x = \frac{1}{2}$, $\cos x = \frac{1}{2}\sqrt{3}$.

Find $\cos 2x$, given the following values of $\sin x$ and $\cos x$:

7. $\sin x = \frac{1}{2}\sqrt{3}$, $\cos x = \frac{1}{2}$. 8. $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$.

Find $\tan 2x$, given the following values of $\tan x$:

9. $\tan x = 0.3673$. 10. $\tan x = 0.2701$.

Find $\cot 2x$, given the following values of $\cot x$ and $\tan x$:

11. $\cot x = 0.3673$. 12. $\tan x = 0.2701$.

Find $\sin 2x$, given the following values of $\sin x$:

13. $\sin x = \frac{5}{13}$. 14. $\sin x = \frac{12}{13}$.

15. As suggested in § 101, find $\sin 3x$ in terms of $\sin x$.

16. As suggested in § 101, find $\cos 3x$ in terms of $\cos x$.

102. Functions of Half an Angle. If we substitute $\frac{1}{2}z$ for x in the formulas $\cos^2 x + \sin^2 x = 1$ (§ 14) and $\cos^2 x - \sin^2 x = \cos 2x$ (§ 101), so as to find the functions of half an angle, we have

$$\cos^2 \tfrac{1}{2}z + \sin^2 \tfrac{1}{2}z = 1,$$

and $$\cos^2 \tfrac{1}{2}z - \sin^2 \tfrac{1}{2}z = \cos z.$$

Subtracting, $$2\sin^2 \tfrac{1}{2}z = 1 - \cos z;$$

whence $$\sin\frac{1}{2}z = \pm\sqrt{\frac{1-\cos z}{2}}.$$

In the above proof, if we add instead of subtract we have

$$2\cos^2 \tfrac{1}{2}z = 1 + \cos z;$$

whence $$\cos\frac{1}{2}z = \pm\sqrt{\frac{1+\cos z}{2}}.$$

Since $\tan\frac{1}{2}z = \dfrac{\sin\frac{1}{2}z}{\cos\frac{1}{2}z}$, and $\cot\frac{1}{2}z = \dfrac{\cos\frac{1}{2}z}{\sin\frac{1}{2}z}$, we have, by dividing,

$$\tan\frac{1}{2}z = \pm\sqrt{\frac{1-\cos z}{1+\cos z}},$$

and $$\cot\frac{1}{2}z = \pm\sqrt{\frac{1+\cos z}{1-\cos z}}.$$

These four formulas are important and should be memorized.

From the formula for $\tan\frac{1}{2}z$ can be derived a formula which is occasionally used in dealing with very small angles. In the triangle ACB we have

$$\tan\tfrac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \pm\sqrt{\frac{1-\frac{b}{c}}{1+\frac{b}{c}}} = \pm\sqrt{\frac{c-b}{c+b}}.$$

Exercise 45. Functions of Half an Angle

Given $\sin 30° = \frac{1}{2}$, find the values of the following:

1. $\sin 15°$. 2. $\cos 15°$. 3. $\tan 15°$. 4. $\cot 15°$. 5. $\cot 7\frac{1}{2}°$.

Given $\tan 45° = 1$, find the values of the following:

6. $\sin 22.5°$. 7. $\cos 22.5°$. 8. $\tan 22.5°$. 9. $\cot 22.5°$. 10. $\cot 11\frac{1}{4}°$.

11. Given $\sin x = 0.2$, find $\sin\frac{1}{2}x$ and $\cos\frac{1}{2}x$.

12. Given $\cos x = 0.7$, find $\sin\frac{1}{2}x$, $\cos\frac{1}{2}x$, $\tan\frac{1}{2}x$, and $\cot\frac{1}{2}x$.

103. Sums and Differences of Functions. Since we have (§§ 92, 97)

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

and $$\sin(x-y) = \sin x \cos y - \cos x \sin y,$$

we find, by addition and subtraction, that

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y,$$

and $$\sin(x+y) - \sin(x-y) = 2 \cos x \sin y.$$

Similarly, by using the formulas for $\cos(x \pm y)$, we obtain

$$\cos(x+y) + \cos(x-y) = 2 \cos x \cos y,$$

and $$\cos(x+y) - \cos(x-y) = -2 \sin x \sin y.$$

By letting $x + y = A$, and $x - y = B$, we have $x = \frac{1}{2}(A+B)$, and $y = \frac{1}{2}(A-B)$, whence

$$\mathbf{\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B),}$$

$$\mathbf{\sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B),}$$

$$\mathbf{\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B),}$$

and $$\mathbf{\cos A - \cos B = -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).}$$

By division we obtain

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \tfrac{1}{2}(A+B) \cot \tfrac{1}{2}(A-B);$$

and since $$\cot \tfrac{1}{2}(A-B) = \frac{1}{\tan \frac{1}{2}(A-B)},$$

we have $$\mathbf{\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.}$$

This is one of the most important formulas in the solution of oblique triangles.

Exercise 46. Formulas

Prove the following formulas:

1. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}.$

2. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}.$

3. $\tan \frac{1}{2} x = \dfrac{\sin x}{1 + \cos x}.$

4. $\cot \frac{1}{2} x = \dfrac{\sin x}{1 - \cos x}.$

If A, B, C are the angles of a triangle, prove that:

5. $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$

6. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C.$

7. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

8. Given $\tan \frac{1}{2} x = 1$, find $\cos x$.

9. Given $\cot \frac{1}{2} x = \sqrt{3}$, find $\sin x$.

10. Prove that $\tan 18° = \dfrac{\sin 33° + \sin 3°}{\cos 33° + \cos 3°}$.

11. Prove that $\sin \frac{1}{2} x \pm \cos \frac{1}{2} x = \sqrt{1 \pm \sin x}$.

12. Prove that $\dfrac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm \tan x \tan y$.

13. Prove that $\tan(45° - x) = \dfrac{1 - \tan x}{1 + \tan x}$.

14. In the triangle ABC prove that

$$\cot \tfrac{1}{2} A + \cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C = \cot \tfrac{1}{2} A \cot \tfrac{1}{2} B \cot \tfrac{1}{2} C.$$

Change to a form involving products instead of sums, and hence more convenient for computation by logarithms:

15. $\cot x + \tan x$.

16. $\cot x - \tan x$.

17. $\cot x + \tan y$.

18. $\cot x - \tan y$.

19. $\dfrac{1 - \cos 2x}{1 + \cos 2x}$.

20. $1 + \tan x \tan y$.

21. $1 - \tan x \tan y$.

22. $\cot x \cot y + 1$.

23. $\cot x \cot y - 1$.

24. $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

25. Prove that $\tan x + \tan y = \dfrac{\sin(x + y)}{\cos x \cos y}$.

26. Prove that $\cot y - \cot x = \dfrac{\sin(x - y)}{\sin x \sin y}$.

27. Given $\tan(x + y) = 3$, and $\tan x = 2$, find $\tan y$.

28. Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x$.

29. Prove that $(\sin x - \cos x)^2 = 1 - \sin 2x$.

30. Prove that $\tan x + \cot x = 2 \csc 2x$.

31. Prove that $\cot x - \tan x = 2 \cos 2x \csc 2x$.

32. Prove that $2 \sin^2(45° - x) = 1 - \sin 2x$.

33. Prove that $\cos 45° + \cos 75° = \cos 15°$.

34. Prove that $1 + \tan x \tan 2x = \tan 2x \cot x - 1$.

Prove the following formulas:

35. $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 2 + 2\cos(x - y)$.

36. $(\sin x + \cos y)^2 + (\sin y + \cos x)^2 = 2 + 2\sin(x + y)$.

37. $\sin(x + y) + \cos(x - y) = (\sin x + \cos x)(\sin y + \cos y)$.

38. $\sin(x + y)\cos y - \cos(x + y)\sin y = \sin x$.

CHAPTER VII

THE OBLIQUE TRIANGLE

104. Geometric Properties of the Triangle. In solving an oblique triangle certain geometric properties are involved in addition to those already mentioned in the preceding chapters, and these should be recalled to mind before undertaking further work with trigonometric functions. These properties are as follows:

The angles opposite the equal sides of an isosceles triangle are equal.

If two angles of a triangle are equal, the sides opposite the equal angles are equal.

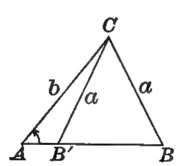
If two angles of a triangle are unequal, the greater side is opposite the greater angle.

If two sides of a triangle are unequal, the greater angle is opposite the greater side.

A triangle is determined, that is, it is completely fixed in form and size, if the following parts are given:

1. *Two sides and the included angle.*
2. *Two angles and the included side.*
3. *Two angles and the side opposite one of them.*
4. *Two sides and the angle opposite one of them.*
5. *Three sides.*

The fourth case, however, will be recalled as the *ambiguous case*, since the triangle is not in general completely determined. If we have given $\angle A$ and sides a and b in this figure, either of the triangles ABC and $AB'C$ will satisfy the conditions.

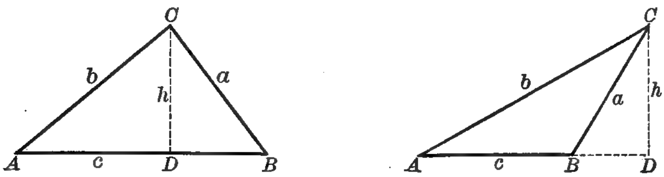


If a is equal to the perpendicular from C on AB, however, the points B and B' will coincide, and hence the two triangles become congruent and the triangle is completely determined.

The five cases relating to the determining of a triangle may be summarized as follows: *A triangle is determined when three independent parts are given.*

This excludes the case of three angles, because they are not independent. That is, $A = 180° - (B + C)$, and therefore A depends upon B and C.

105. Law of Sines. In the triangle ABC, using either of the figures as here shown, we have the following relations.



In either figure, $\dfrac{h}{b} = \sin A.$

In the first figure, $\dfrac{h}{a} = \sin B,$

and in the second figure, $\dfrac{h}{a} = \sin(180° - B)$

$$= \sin B.$$

Therefore, whether h lies within or without the triangle, we obtain, by division, the following relation:

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

In the same way, by drawing perpendiculars from the vertices A and B to the opposite sides, we may obtain the following relations:

$$\frac{b}{c} = \frac{\sin B}{\sin C},$$

and

$$\frac{a}{c} = \frac{\sin A}{\sin C}.$$

This relation between the sides and the sines of the opposite angles is called the Law of Sines and may be expressed as follows:

The sides of a triangle are proportional to the sines of the opposite angles.

If we multiply $\frac{a}{b} = \frac{\sin A}{\sin B}$ by b, and divide by $\sin A$, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Similarly, we may obtain the following:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

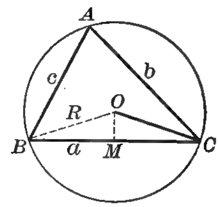
and this is frequently given as the Law of Sines.

It is also apparent that $a \sin B = b \sin A$, $a \sin C = c \sin A$, and $b \sin C = c \sin B$, three relations which are still another form of the Law of Sines.

106. The Law of Sines extended. There is an interesting extension of the Law of Sines with respect to the diameter of the circle circumscribed about a triangle.

Circumscribe a circle about the triangle ABC and draw the radii OB, OC, as shown in the figure. Let R denote the radius. Draw OM perpendicular to BC. Since the angle BOC is a central angle intercepting the same arc as the angle A, the angle $BOC = 2A$; hence the angle $BOM = A$; then

$$BM = R \sin BOM = R \sin A.$$



Therefore $\qquad a = 2R \sin A.$

In like manner, $\qquad b = 2R \sin B,$

and $\qquad c = 2R \sin C.$

Therefore $\qquad 2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}.$

That is, *The ratio of any side of a triangle to the sine of the opposite angle is numerically equal to the diameter of the circumscribed circle.*

Exercise 47. Law of Sines

1. Consider the formula $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$ when $B = 90°$; when $A = 90°$; when $A = B$; when $a = b$.

2. Prove by the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

3. Prove Ex. 2 for the bisector of an exterior angle of a triangle.

4. The triangle ABC has $A = 78°$, $B = 72°$, and $c = 4$ in. Find the diameter of the circumscribed circle.

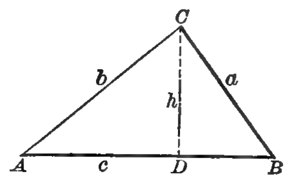
5. The triangle ABC has $A = 76° 37'$, $B = 81° 46'$, and $c = 368.4$ ft. Find the diameter of the circumscribed circle.

6. What is the diameter of the circle circumscribed about an equilateral triangle of side 7.4 in.? What is the diameter of the circle inscribed in the same triangle?

7. What is the diameter of the circle circumscribed about an isosceles triangle of base 4.8 in. and vertical angle 10°?

8. What is the diameter of the circle circumscribed about an isosceles triangle whose vertical angle is 18° and the sum of the two equal sides 18 in.?

107. Applications of the Law of Sines. If we have given any side of a triangle, and any two of the angles, we are able to solve the triangle by means of the Law of Sines. Thus, if we have given a, A, and B, in this triangle, we can find the remaining parts as follows:



1. $C = 180° - (A + B)$.

2. $\frac{b}{a} = \frac{\sin B}{\sin A}$;

$$\therefore b = \frac{a \sin B}{\sin A} = \frac{a}{\sin A} \times \sin B.$$

3. $\frac{c}{a} = \frac{\sin C}{\sin A}$; $\quad \therefore c = \frac{a \sin C}{\sin A} = \frac{a}{\sin A} \times \sin C$.

For example, given $a = 24.31$, $A = 45° 18'$, and $B = 22° 11'$, solve the triangle.

The work may be arranged as follows:

$a = 24.31$	$\log a = 1.38578$	$= 1.38578$
$A = 45° 18'$	$\text{colog} \sin A = 0.14825$	$= 0.14825$
$B = 22° 11'$	$\log \sin B = 9.57700$	$\log \sin C = 9.96556$
$A + B = 67° 29'$	$\log b = 1.11103$	$\log c = 1.49959$
$\therefore C = 112° 31'$	$\therefore b = 12.913$	$\therefore c = 31.593$

When -10 is omitted after a logarithm or cologarithm to which it belongs, it must still be remembered that the logarithm or cologarithm is 10 too large.

The length of a having been given only to four significant figures, the values of b and c are to be depended upon only to the same number of significant figures in practical measurement. In the above example a is given to only four significant figures, and hence we say that $b = 12.91$, and $c = 31.59$.

Exercise 48. Law of Sines

Solve the triangle ABC, given the following parts:

1. $a = 500$, $A = 10° 12'$, $B = 46° 36'$.
2. $a = 795$, $A = 79° 59'$, $B = 44° 41'$.
3. $a = 804$, $A = 99° 55'$, $B = 45° 1'$.
4. $a = 820$, $A = 12° 49'$, $B = 141° 59'$.
5. $c = 1005$, $A = 78° 19'$, $B = 54° 27'$.
6. $b = 13.57$, $B = 13° 57'$, $C = 57° 13'$.
7. $a = 6412$, $A = 70° 55'$, $C = 52° 9'$.
8. $b = 999$, $A = 37° 58'$, $C = 65° 2'$.

Solve Exs. 9–14 without using logarithms:

9. Given $b = 7.071$, $A = 30°$, and $C = 105°$, find a and c.

10. Given $c = 9.562$, $A = 45°$, and $B = 60°$, find a and b.

11. The base of a triangle is 600 ft. and the angles at the base are 30° and 120°. Find the other sides and the altitude.

12. Two angles of a triangle are 20° and 40°. Find the ratio of the opposite sides.

13. The angles of a triangle are as $5:10:21$, and the side opposite the smallest angle is 3. Find the other sides.

14. Given one side of a triangle 27 in., and the adjacent angles each equal to 30°, find the radius of the circumscribed circle.

15. The angles B and C of a triangle ABC are $50°\,30'$ and $122°\,9'$ respectively, and BC is 9 mi. Find AB and AC.

16. In a parallelogram, given a diagonal d and the angles x and y which this diagonal makes with the sides, find the sides. Compute the results when $d = 11.2$, $x = 19°\,1'$, and $y = 42°\,54'$.

17. A lighthouse was observed from a ship to bear N. 34° E.; after the ship sailed due south 3 mi. the lighthouse bore N. 23° E. Find the distance from the lighthouse to the ship in each position.

The phrase *to bear N. 34° E.* means that the line of sight to the lighthouse is in the northeast quarter of the horizon and makes, with a line due north, an angle of 34°.

18. A headland was observed from a ship to bear directly east; after the ship had sailed 5 mi. N. 31° E. the headland bore S. 42° E. Find the distance from the headland to the ship in each position.

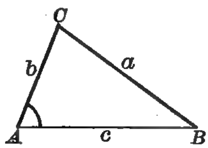
19. In a trapezoid, given the parallel sides a and b, and the angles x and y at the ends of one of the parallel sides, find the nonparallel sides. Compute the results when $a = 15$, $b = 7$, $x = 70°$, $y = 40°$.

20. Two observers 5 mi. apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58° respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

21. A balloon is directly above a straight road $7\frac{1}{4}$ mi. long, joining two towns. The balloonist observes that the first town makes an angle of 42° and the second town an angle of 38° with the perpendicular. Find the distance from the balloon to each town, and also the height of the balloon above the plain.

108. The Ambiguous Case. As mentioned in § 104, if two sides of a triangle and the angle opposite one of them are given, the solution will lead, in general, to two triangles. Thus, if we have the two sides a and b and the angle A given, we proceed to solve the triangle as follows:

$$C = 180^\circ - (A + B);$$



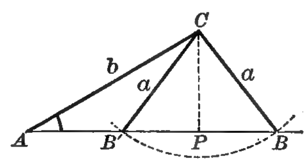
hence we can find C if we can find B.

Furthermore, $$\frac{c}{a} = \frac{\sin C}{\sin A},$$

whence $$c = \frac{a \sin C}{\sin A};$$

hence we can find c if we can find C, and we can also find c if we can find B. But to find B we have

$$\frac{\sin B}{\sin A} = \frac{b}{a},$$



whence $$\sin B = \frac{b \sin A}{a}.$$

Therefore we do not find B directly, but only $\sin B$. But when an angle is determined by its sine, it admits of two values which are supplements of each other (§ 86); hence either of the two values of B may be taken unless one of them is excluded by the conditions of the problem.

In general, therefore, either of the triangles ABC and $AB'C$ fulfills the given conditions.

Exercise 49. The Ambiguous Case

In the triangle ABC given a, b, and A, prove that:

1. If $a > b$, then $A > B$, B is acute, and there is one and only one triangle which will satisfy the given conditions.

2. If $a = b$, both A and B are acute, and there is one and only one triangle which will satisfy the given conditions, and this triangle is isosceles.

3. If $a < b$, then A must be acute to have the triangle possible, and there are then two triangles which will satisfy the given conditions.

4. If $a = b \sin A$, the required triangle is a right triangle.

5. If $a < b \sin A$, the triangle is impossible.

6. If $A = B$, there is one, and only one, triangle.

109. Number of Solutions to be expected. We may summarize the results found on page 112 as follows:

There are two solutions if A is acute and the value of a lies between b and $b \sin A$.

There is no solution if A is acute and $a < b \sin A$; or if A is obtuse and $a < b$, or $a = b$.

There is one solution in each of the other cases.

The number of solutions can often be determined by inspection. In case of doubt, find the value of $b \sin A$.

We can also determine the number of solutions by considering the value of $\log \sin B$. If $\log \sin B = 0$, then $\sin B = 1$ and $B = 90°$. Therefore the triangle required is a right triangle. If $\log \sin B > 0$, then $\sin B > 1$, and hence the triangle is impossible. If $\log \sin B < 0$, there is *one solution* when $a > b$; there are *two solutions* when $a < b$.

When there are two solutions, let B', C', c', denote the unknown parts of the second triangle; then

$$B' = 180° - B,$$

$$C' = 180° - (A + B') = B - A,$$

and

$$c' = \frac{a \sin C'}{\sin A}.$$

110. Illustrative Problems. The following may be taken as illustrative of the above cases:

1. Given $a = 16$, $b = 20$, and $A = 106°$, find the remaining parts.

In this case $a < b$ and $A > 90°$. Since $a < b$, it follows that $A < B$. Hence if $A > 90°$, B must also be greater than 90°. But a triangle cannot have two obtuse angles. Therefore the triangle is impossible.

2. Given $a = 36$, $b = 80$, and $A = 30°$, find the remaining parts.

Here we have $b \sin A = 80 \times \frac{1}{2} = 40$; so that $a < b \sin A$ and the triangle is impossible. Draw the figure to illustrate this fact.

3. Given $a = 25$, $b = 50$, and $A = 30°$, find the remaining parts.

Here we have $b \sin A = 50 \times \frac{1}{2} = 25$; but a is also equal to 25. Hence B must be a right angle. ABC is therefore a right triangle and there is only one solution.

4. Given $a = 30$, $b = 30$, and $A = 60°$, find the remaining parts.

Here we have $a = b$, and A an acute angle. Hence there is one solution and only one. It is evident, also, that the triangle is not only isosceles but equilateral.

5. Given $a = 3.4$, $b = 3.4$, and $A = 45°$, find the remaining parts.

Here we have $a = b$, and A an acute angle. Hence there is one solution and only one. It is evident, also, that the triangle is not only isosceles but right.

6. Given $a = 72{,}630$, $b = 117{,}480$, and $A = 80° \, 0' \, 50''$, find B, C, and c.

$$\begin{aligned} \log b &= 5.06997 \\ \log \sin A &= 9.99337 \\ \operatorname{colog} a &= \underline{5.13888} \\ \log \sin B &= 0.20222 \end{aligned}$$

Here $\log \sin B > 0$.
Therefore $\sin B > 1$, which is impossible.

Therefore there is *no solution.*

7. Given $a = 13.2$, $b = 15.7$, and $A = 57° \, 13' \, 15''$, find B, C, and c.

$$\begin{aligned} \log b &= 1.19590 \\ \log \sin A &= 9.92467 \\ \operatorname{colog} a &= \underline{8.87943} \\ \log \sin B &= 0.00000 \\ \therefore B &= 90° \\ \therefore C &= 32° \, 46' \, 45'' \end{aligned}$$

$$\begin{aligned} c &= b \cos A \\ \log b &= 1.19590 \\ \log \cos A &= \underline{9.73352} \\ \log c &= 0.92942 \\ \therefore c &= 8.5 \end{aligned}$$

Therefore there is *one solution.*
Since $B = 90°$, the triangle is a right triangle.

8. Given $a = 767$, $b = 242$, and $A = 36° \, 53' \, 2''$, find B, C, and c.

$$\begin{aligned} \log b &= 2.38382 \\ \log \sin A &= 9.77830 \\ \operatorname{colog} a &= \underline{7.11520} \\ \log \sin B &= 9.27732 \\ \therefore B &= 10° \, 54' \, 58'' \\ \therefore C &= 132° \, 12' \, 0'' \end{aligned}$$

$$\begin{aligned} \log a &= 2.88480 \\ \log \sin C &= 9.86970 \\ \operatorname{colog} \sin A &= \underline{0.22170} \\ \log c &= 2.97620 \\ \therefore c &= 946.68 \\ &= 946.7 \end{aligned}$$

Here $a > b$, and $\log \sin B < 0$.
Therefore there is *one solution.*

9. Given $a = 177.01$, $b = 216.45$, and $A = 35° \, 36' \, 20''$, find the other parts.

$$\begin{aligned} \log b &= 2.33536 \\ \log \sin A &= 9.76507 \\ \operatorname{colog} a &= \underline{7.75200} \\ \log \sin B &= 9.85243 \\ \therefore B &= 45° \, 23' \, 28'' \text{ or} \\ & 134° \, 36' \, 32'' \\ \therefore C &= 99° \, 0' \, 12'' \text{ or} \\ & 9° \, 47' \, 8'' \end{aligned}$$

$$\begin{array}{r|l} \log a = 2.24800 & 2.24800 \\ \log \sin C = 9.99462 & 9.23035 \\ \operatorname{colog} \sin A = \underline{0.23493} & \underline{0.23493} \\ \log c = 2.47755 & 1.71328 \end{array}$$

$$\begin{aligned} \therefore c &= 300.29 \text{ or } 51.675 \\ &= 300.29 \text{ or } 51.68 \end{aligned}$$

Here $a < b$, and $\log \sin B < 0$.
Therefore there are *two solutions.*

Exercise 50. The Oblique Triangle

Find the number of solutions, given the following:

1. $a = 80$, $b = 100$, $A = 30°$.
2. $a = 50$, $b = 100$, $A = 30°$.
3. $a = 40$, $b = 100$, $A = 30°$.
4. $a = 100$, $b = 100$, $A = 30°$.
5. $a = 13.4$, $b = 11.46$, $A = 77° 20'$.
6. $a = 70$, $b = 75$, $A = 60°$.
7. $a = 134.16$, $b = 84.54$, $B = 52° 9'$.
8. $a = 200$, $b = 100$, $A = 30°$.

Solve the triangles, given the following:

9. $a = 840$, $b = 485$, $A = 21° 31'$.
10. $a = 9.399$, $b = 9.197$, $A = 120° 35'$.
11. $a = 91.06$, $b = 77.04$, $A = 51° 9'$.
12. $a = 55.55$, $b = 66.66$, $B = 77° 44'$.
13. $a = 309$, $b = 360$, $A = 21° 14'$.
14. $a = 34$, $b = 22$, $B = 30° 20'$.
15. $b = 19$, $c = 18$, $C = 15° 49'$.
16. $a = 8.716$, $b = 9.787$, $A = 38° 14' 12''$.
17. $a = 4.4$, $b = 5.21$, $A = 57° 37' 17''$.

18. Given $a = 75$, $b = 29$, and $B = 16° 15'$, find the difference between the areas of the two triangles which meet these conditions.

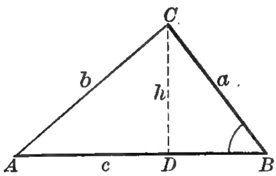
19. In a parallelogram, given the side a, a diagonal d, and the angle A made by the two diagonals, find the other diagonal. As a special case consider the parallelogram in which $a = 35$, $d = 63$, and $A = 21° 36'$.

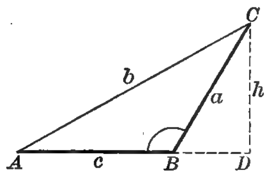
20. In a parallelogram $ABCD$, given $AD = 3$ in., $BD = 2.5$ in., and $A = 47° 20'$, find AB.

21. In a quadrilateral $ABCD$, given $AC = 4$ in., $\angle BAC = 35°$, $\angle B = 75° 20'$, $\angle D = 38° 30'$, and $\angle BAD = 70° 40'$, find the length of each of the four sides.

22. In a pentagon $ABCDE$, given $\angle A = 110° 50'$, $\angle B = 106° 30'$, $\angle E = 104° 10'$, $\angle BAC = 30°$, $\angle DAE = 34° 56'$, $\angle ADC = 52° 30'$, and $AC = 6$ in., find the sides and the remaining angles of the pentagon.

111. Law of Cosines. This law gives the value of one side of a triangle in terms of the other two sides and the angle included between them.





In either figure, $a^2 = h^2 + \overline{BD}^2.$

In the first figure, $BD = c - AD.$

In the second figure, $BD = AD - c.$

In either case, $\overline{BD}^2 = \overline{AD}^2 - 2c \times AD + c^2.$

Therefore, in all cases, $a^2 = h^2 + \overline{AD}^2 + c^2 - 2c \times AD.$

Now $h^2 + \overline{AD}^2 = b^2,$

and $AD = b \cos A.$

Therefore $\boldsymbol{a^2 = b^2 + c^2 - 2\,bc \cos A.}$

In like manner it may be proved that

$$b^2 = c^2 + a^2 - 2\,ca \cos B,$$

and

$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

The three formulas have precisely the same form, and the Law of Cosines may be stated as follows:

The square on any side of a triangle is equal to the sum of the squares on the other two sides diminished by twice their product into the cosine of the included angle.

It will be seen that if $A = 90°$, we have

$$a^2 = b^2 + c^2 - 2\,bc \cos 90°$$
$$= b^2 + c^2.$$

In other words we have the Pythagorean Theorem as a special case. Hence this is sometimes called the *Generalized Pythagorean Theorem.*

It will also be seen that the law includes two other familiar propositions of geometry, one of which is the following:

In an obtuse triangle the square on the side opposite the obtuse angle is equivalent to the sum of the squares on the other two sides increased by twice the product of one of those sides by the projection of the other upon that side.

This and the analogous proposition are given as exercises on page 117.

Exercise 51. Law of Cosines

1. Using the figures on page 116, prove that, whether the angle B is acute or obtuse, $c = a \cos B + b \cos A$.

2. What are the two symmetrical formulas obtained by changing the letters in Ex. 1? What does the formula in Ex. 1 become when $B = 90°$?

3. Show that the sum of the squares on the sides of a triangle is equal to $2(ab \cos C + bc \cos A + ca \cos B)$.

4. Consider the Law of Cosines in the case of the triangle $a = 5$, $b = 12$, $c = 6$.

5. Given $a = 5$, $b = 5$, and $C = 60°$, find c.

6. Given $a = 10$, $b = 10$, and $C = 45°$, find c

7. Given $a = 8$, $b = 5$, and $C = 60°$, find c.

8. From the formula $a^2 = b^2 + c^2 - 2\,bc \cos A$ deduce a formula for $\cos A$. From this result find the value of A when $b^2 + c^2 = a^2$.

9. Prove that if $\dfrac{\cos A}{b} = \dfrac{\cos B}{a}$ the triangle is either isosceles or right.

10. Prove that $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2\,abc}$.

11. Prove that $\dfrac{b^2}{a} \cos A + \dfrac{c^2}{b} \cos B + \dfrac{a^2}{c} \cos C = \dfrac{a^4 + b^4 + c^4}{2\,abc}$.

12. From the Law of Cosines prove that the square on the side opposite an acute angle of a triangle is equal to the sum of the squares on the other two sides minus twice the product of either side and the projection of the other side upon it.

13. As in Ex. 12, consider the geometric proposition relating to the square on the side opposite an obtuse angle.

14. In the parallelogram $ABCD$, given $AB = 4$ in., $AD = 5$ in., and $A = 38° 40'$, find the two diagonals.

15. In the parallelogram $ABCD$, given $AB = 7$ in., $AC = 10$ in., and $\angle BAC = 36° 7'$, find the side BC and the diagonal BD.

16. In the quadrilateral $ABCD$, given $AC = 3.6$ in., $AD = 4$ in., $BC = 2.4$ in., $\angle ACB = 29° 40'$, and $\angle CAD = 71° 20'$, find the other two sides and all four angles of the quadrilateral.

17. In the pentagon $ABCDE$, given $AB = 3.4$ in., $AC = 4.1$ in., $AD = 3.9$ in., $AE = 2.2$ in., $\angle BAC = 38° 7'$, $\angle CAD = 41° 22'$, and $\angle DAE = 32° 5'$, find the perimeter of the pentagon.

112. Law of Tangents. Since $\dfrac{a}{b}=\dfrac{\sin A}{\sin B}$, by the Law of Sines, it follows by the theory of proportion that

$$\frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B}.$$

This is easily seen without resorting to the theory of proportion. For, since $a\sin B=b\sin A$ (§ 105), we have

$$a\sin B-b\sin A=b\sin A-a\sin B$$

Adding, $$a\sin A-b\sin B=a\sin A-b\sin B$$

$$a\sin A+a\sin B-b\sin A-b\sin B=a\sin A-a\sin B+b\sin A-b\sin B,$$

or $$(a-b)(\sin A+\sin B)=(a+b)(\sin A-\sin B);$$

whence, by division, $$\frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B}.$$

But by § 103, $$\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Therefore $$\boldsymbol{\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}}.$$

By merely changing the letters,

$$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)},$$

and $$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

Hence the Law of Tangents:

The difference between two sides of a triangle is to their sum as the tangent of half the difference between the opposite angles is to the tangent of half their sum.

In the case of a triangle, if we know the two sides a and b and the included angle C, we have our choice of two methods of solving. From the Law of Cosines we can find c, and then, from the Law of Sines, we can find A and B. Or we can find $A+B$ by taking C from 180°, and then, since we also know $a+b$ and $a-b$, we can find $A-B$. From $A+B$ and $A-B$ we can find A and B. This second method is usually the simpler one.

If $b>a$, then $B>A$. The formula is still true, but to avoid negative numbers the formula in this case should be written

$$\frac{b-a}{b+a}=\frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)}.$$

Exercise 52. Law of Tangents

Find the form to which $\dfrac{a-b}{a+b}=\dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$ *reduces when:*

1. $C=90°$.
2. $a=b$.
3. $A=B=C$.
4. $A-B=90°$, and $B=C$.

Prove the following formulas:

5. $\dfrac{b-c}{b+c}=\tan\frac{1}{2}(B-C)\cot\frac{1}{2}(B+C)$.

6. $\tan\frac{1}{2}(B-C)=\dfrac{b-c}{b+c}\cot\frac{1}{2}A$.

7. $\dfrac{a+b}{a-b}=\dfrac{\cot\frac{1}{2}(A-B)}{\cot\frac{1}{2}(A+B)}$.

8. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$.

9. $\dfrac{\sin B+\sin C}{\sin B-\sin C}=\dfrac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C)}$.

10. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\tan\frac{1}{2}(A+B)\cot\frac{1}{2}(A-B)$.

11. To what does the formula in Ex. 8 reduce when $A=B$?

12. To what does the formula in Ex. 9 reduce when $B=C=60°$?

13. To what does the formula in Ex. 10 reduce when the triangle is equilateral?

14. To what does the Law of Tangents, in the form stated at the top of this page, reduce in the case of an isosceles triangle in which $a=b$? What does this prove with respect to the angles opposite the equal sides?

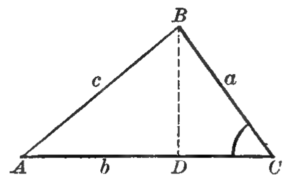
15. By the help of the Law of Tangents prove that an equilateral triangle is also equiangular.

16. By the help of the Law of Tangents prove that an equiangular triangle is also equilateral.

17. Given any three sides and any three angles of a quadrilateral, show how the fourth side and the fourth angle can be found. Show also that it is not necessary to have so many parts given, and find the smallest number of parts that will solve the quadrilateral.

18. What sides, what diagonals, and what angles of a pentagon is it necessary to know in order, by the aid of the Law of Tangents alone, to solve the pentagon?

113. Applications to Triangles. The Law of Cosines and the Law of Tangents are frequently used in the solution of triangles. This is particularly the case when we have given two sides, a and b, and the included angle C.



There are two convenient ways of finding the angles A and B, the first being by the Law of Tangents. This law may be written

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \times \tan \tfrac{1}{2}(A+B).$$

Since $\frac{1}{2}(A+B) = \frac{1}{2}(180° - C)$, the value of $\frac{1}{2}(A+B)$ is known, so that this equation enables us to find the value of $\frac{1}{2}(A-B)$. We then have

$$\tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B) = A,$$

and

$$\tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B) = B.$$

The second method of finding A and B is as follows: In the above figure let BD be perpendicular to AC.

Then
$$\tan A = \frac{BD}{AD} = \frac{BD}{AC - DC}.$$

Now
$$BD = a \sin C,$$

and
$$DC = a \cos C.$$

$$\therefore \tan A = \frac{a \sin C}{b - a \cos C}.$$

Since A and C are now known, B can be found.

This is not so convenient as the first method, because it is not so well adapted to work with logarithms.

The side c may now be found by the Law of Sines, thus:

$$c = \frac{a \sin C}{\sin A}, \quad \text{or} \quad c = \frac{b \sin C}{\sin B}.$$

Instead of finding A and B first, and from these values finding c, we may first find c and then find A and B. To find c first we may write the Law of Cosines (§ 111) as follows:

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}.$$

Having thus found c, and already knowing a, b, and C, we have

$$\sin A = \frac{a \sin C}{c}, \quad \sin B = \frac{b \sin C}{c}.$$

In general this is not so convenient as the first method given above, because the formula for c is not so well adapted to work with logarithms.

114. Illustrative Problems. 1. Given $C = 63° 35' 30''$, $a = 748$, and $b = 375$, find A, B, and c.

We see that $a + b = 1123$, $a - b = 373$, and $A + B = 180° - C = 116° 24' 30''$. Hence $\frac{1}{2}(A + B) = 58° 12' 15''$.

$\log(a - b) = 2.57171$	$\log b = 2.57403$
$\text{colog}(a + b) = 6.94962$	$\log \sin C = 9.95214$
$\log \tan \frac{1}{2}(A + B) = 0.20766$	$\text{colog} \sin B = 0.30073$
$\log \tan \frac{1}{2}(A - B) = 9.72899$	$\log c = 2.82690$
$\therefore \frac{1}{2}(A - B) = 28° 10' 54''$	$\therefore c = 671.27$

After finding $\frac{1}{2}(A - B)$ we combine this with $\frac{1}{2}(A + B)$ and find $A = 86° 23' 9''$ and $B = 30° 1' 21''$.

In the above example, in finding the side c we use the angle B rather than the angle A, because A is near $90°$. The use of the sine of an angle near $90°$ should be avoided, because it varies so slowly that we cannot determine the angle accurately when the sine is given.

2. Given $a = 4$, $c = 6$, and $B = 60°$, find the third side b.

Here the Law of Cosines may be used to advantage, because the numbers are so small as to make the computation easy. We have

$$b = \sqrt{a^2 + c^2 - 2ac \cos B} = \sqrt{16 + 36 - 24} = \sqrt{28};$$

$$\log 28 = 1.44716, \qquad \log \sqrt{28} = 0.72358, \qquad \sqrt{28} = 5.2915;$$

that is, to three significant figures, $\quad b = 5.292$.

Exercise 53. Solving Triangles

Solve these triangles, given the following parts:

1. $a = 77.99$, $\quad b = 83.39$, $\quad C = 72° 15'$.
2. $b = 872.5$, $\quad c = 632.7$, $\quad A = 80°$.
3. $a = 17$, $\quad b = 12$, $\quad C = 59° 17'$.
4. $b = \sqrt{5}$, $\quad c = \sqrt{3}$, $\quad A = 35° 53'$.
5. $a = 0.917$, $\quad b = 0.312$, $\quad C = 33° 7' 9''$.
6. $a = 13.715$, $\quad c = 11.214$, $\quad B = 15° 22' 36''$.
7. $b = 3000.9$, $\quad c = 1587.2$, $\quad A = 86° 4' 4''$.
8. $a = 4527$, $\quad b = 3465$, $\quad C = 66° 6' 27''$.
9. $a = 55.14$, $\quad b = 33.09$, $\quad C = 30° 24'$.
10. $a = 47.99$, $\quad b = 33.14$, $\quad C = 175° 19' 10''$.
11. $a = 210$, $\quad b = 105$, $\quad C = 36° 52' 12''$.
12. $a = 100$, $\quad b = 900$, $\quad C = 65°$.

Solve these triangles, given the following parts:

13. $a = 409$, $b = 169$, $C = 117.7°$.
14. $a = 6.25$, $b = 5.05$, $C = 105.77°$.
15. $a = 3718$, $b = 1507$, $C = 95.86°$.
16. $a = 46.07$, $b = 22.29$, $C = 66.36°$.
17. $b = 445$, $c = 624$, $A = 10.88°$.
18. $b = 15.7$, $c = 43.6$, $A = 57.22°$.

19. If two sides of a triangle are each equal to 6, and the included angle is 60°, find the third side by two different methods.

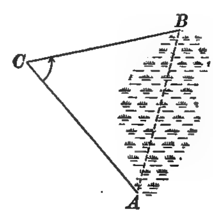
20. If two sides of a triangle are each equal to 6, and the included angle is 120°, find the third side by three different methods.

21. Apply the first method given on page 120 to the case in which a is equal to b; that is, the case in which the triangle is isosceles.

22. If two sides of a triangle are 10 and 11, and the included angle is 50°, find the third side.

23. If two sides of a triangle are 43.301 and 25, and the included angle is 30°, find the third side.

24. In order to find the distance between two objects, A and B, separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yd., $CB = 3475.6$ yd., together with the angle $ACB = 62° 31'$, were measured. Find the distance from A to B.



25. Two inaccessible objects, A and B, are each viewed from two stations, C and D, on the same side of AB and 562 yd. apart. The angle ACB is 62° 12′, BCD 41° 8′, ADB 60° 49′, and ADC 34° 51′. Required the distance AB.

26. In order to find the distance between two objects, A and B, separated by a pond, a station C was chosen, and it was found that $CA = 426$ yd., $CB = 322.4$ yd., and $ACB = 68° 42'$. Required the distance from A to B.

27. Two trains start at the same time from the same station and move along straight tracks that form an angle of 30°, one train at the rate of 30 mi. an hour, the other at the rate of 40 mi. an hour. How far apart are the trains at the end of half an hour?

28. In a parallelogram, given the two diagonals 5 and 6 and the angle which they form 49° 18′, find the sides.

115. Given the Three Sides. Given the three sides of a triangle, it is possible to find the angles by the Law of Cosines. Thus, from

$$a^2 = b^2 + c^2 - 2\,bc\cos A,$$

we have
$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

This formula is not, however, adapted to work with logarithms. In order to remedy this difficulty we shall now proceed to change its form.

Let s equal the semiperimeter of the triangle; that is,

let $\quad a + b + c = 2\,s.$

Then $\quad b + c - a = 2\,s - 2\,a = 2(s - a),$

$\quad c + a - b = 2(s - b),$

and $\quad a + b - c = 2(s - c).$

Hence
$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc}$$
$$= \frac{a^2 - (b - c)^2}{2\,bc} = \frac{(a + b - c)(a - b + c)}{2\,bc}$$
$$= \frac{2(s - b)(s - c)}{bc}.$$

In the same way the value of $1 + \cos A$ is

$$1 + \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{2\,bc + b^2 + c^2 - a^2}{2\,bc} = \frac{(b + c)^2 - a^2}{2\,bc}$$
$$= \frac{(b + c + a)(b + c - a)}{2\,bc} = \frac{2\,s(s - a)}{bc}.$$

But from § 102 we know that

$$1 - \cos A = 2\sin^2\tfrac{1}{2}A, \qquad \text{and} \qquad 1 + \cos A = 2\cos^2\tfrac{1}{2}A.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}A = \frac{2(s - b)(s - c)}{bc}, \text{ and } 2\cos^2\tfrac{1}{2}A = \frac{2\,s(s - a)}{bc}.$$

It therefore follows that

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}},$$

and
$$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s - a)}{bc}}.$$

Furthermore, since $\tan x = \dfrac{\sin x}{\cos x}$, we have

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

By merely changing the letters in the formulas given on page 123, we have the following:

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}}, \qquad \sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}},$$

$$\cos \tfrac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}, \qquad \cos \tfrac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}},$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \qquad \tan \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

There is then a choice of three different formulas for finding the value of each angle. If half the angle is very near 0°, the formula for the cosine will not give a very accurate result, because the cosines of angles near 0° differ little in value; and the same is true of the formula for the sine when half the angle is very near 90°. Hence in the first case the formula for the sine, and in the second that for the cosine, should be used.

But in general the formulas for the tangent are to be preferred, the tangent as a rule changing more rapidly than the sine or cosine.

It is not necessary to compute by the formulas more than two angles, for the third may then be found from the equation $A + B + C = 180°$. There is this advantage, however, in computing all three angles by the formulas, that we may then use the sum of the angles as a test of the accuracy of the results.

116. Checks on the Angles. In case it is desired to compute all the angles for the purpose of checking the work, the formulas for the tangent may be put in a more convenient form.

The formula for $\tan \frac{1}{2} A$ may be written thus:

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$$

$$= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Hence, if we put
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

we have
$$\tan \tfrac{1}{2} A = \frac{r}{s-a}.$$

Likewise,
$$\tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

For example, if $a = 3$, $b = 3.5$, and $c = 4.5$, we have $s = 5.5$, $s - a = 2.5$, $s - b = 2$, and $s - c = 1$.

$$\therefore r = \sqrt{\frac{2.5 \times 2 \times 1}{5.5}} = \sqrt{\frac{5}{5.5}} = \sqrt{\frac{10}{11}} = 0.9534.$$

$$\therefore \tan \tfrac{1}{2} A = 0.9534 \div 2.5 = 0.3814.$$

$$\therefore \tfrac{1}{2} A = 20° \, 53'.$$

$$\therefore A = 41° \, 46'.$$

Exercise 54. Formulas of the Triangle

1. Given $\tan\frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, express the value of $\log\tan\frac{1}{2}A$.

2. Given $\sin\frac{1}{2}A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, express the value of $\log\sin\frac{1}{2}A$.

3. Given $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$, express the value of $\log r$.

4. Given $\tan\frac{1}{2}A = \dfrac{r}{s-a}$, express the value of $\log\tan\frac{1}{2}A$.

5. Given $\tan\frac{1}{2}A = \dfrac{r}{s-a}$, express the value of $\log r$.

6. Of the three values for $\tan\frac{1}{2}A$,

$$\sqrt{\frac{1-\cos A}{1+\cos A}}, \qquad (\S\ 102)$$

$$\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \qquad (\S\ 115)$$

and

$$\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \qquad (\S\ 116)$$

which is the easiest to treat by logarithms? Express the logarithms of the results and show why your answer is correct.

7. Given $a = 4$, $b = 5$, and $c = 6$, find the value of $\tan\frac{1}{2}A$, and then find the value of A.

8. Deduce the equation

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

from the equation

$$\tan\tfrac{1}{2}A = \sqrt{\frac{1-\cos A}{1+\cos A}}.$$

9. Discuss the formula

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

for the case of an equilateral triangle, say when $a = 4$.

117. Illustrative Problems. 1. Given $a = 3.41$, $b = 2.60$, $c = 1.58$, find the angles.

Since it is given that $a = 3.41$, $b = 2.60$, and $c = 1.58$, it follows that $2s = 7.59$ and $s = 3.795$. Therefore

$$s - a = 0.385, \qquad s - b = 1.195, \qquad s - c = 2.215.$$

Using the formula of § 115 and the corresponding formula for $\tan \frac{1}{2} B$, we may arrange the work as follows:

$$\begin{aligned}
\operatorname{colog} s &= 9.42079 \\
\operatorname{colog}(s-a) &= 0.41454 \\
\log(s-b) &= 0.07737 \\
\log(s-c) &= 0.34537 \\
2\,)\,&\overline{0.25807} \\
\log\tan\tfrac{1}{2}A &= 0.12903 \\
\therefore\ \tfrac{1}{2}A &= 53°\ 23'\ 20'' \\
\therefore\ A &= 106°\ 46'\ 40''
\end{aligned}
\qquad\qquad
\begin{aligned}
\operatorname{colog} s &= 9.42079 - 10 \\
\log(s-a) &= 9.58546 - 10 \\
\operatorname{colog}(s-b) &= 9.92263 - 10 \\
\log(s-c) &= 0.34537 \\
2\,)\,&\overline{19.27425 - 20} \\
\log\tan\tfrac{1}{2}B &= 9.63713 - 10 \\
\therefore\ \tfrac{1}{2}B &= 23°\ 26'\ 37'' \\
\therefore\ B &= 46°\ 53'\ 14''
\end{aligned}$$

$$\therefore\ A + B = 153°\ 39'\ 54'', \text{ and } C = 26°\ 20'\ 6''.$$

2. Solve the above problem by finding all three angles by the use of the formulas on page 124.

Since it is given that $a = 3.41$, $b = 2.60$, and $c = 1.58$, it follows that $2s = 7.59$ and $s = 3.795$. Therefore

$$s - a = 0.385, \qquad s - b = 1.195, \qquad s - c = 2.215.$$

Here the work may be compactly arranged as follows, if we find $\log\tan\frac{1}{2}A$, etc., by *subtracting* $\log(s-a)$, etc., from $\log r$ instead of adding the cologarithm.

$$\begin{aligned}
\log(s-a) &= 9.58546 \\
\log(s-b) &= 0.07737 \\
\log(s-c) &= 0.34537 \\
\operatorname{colog} s &= \underline{9.42079} \\
\log r^2 &= 9.42899 \\
\log r &= 9.71450
\end{aligned}
\qquad\qquad
\begin{aligned}
\log\tan\tfrac{1}{2}A &= 10.12903 \\
\log\tan\tfrac{1}{2}B &= 9.63713 \\
\log\tan\tfrac{1}{2}C &= \underline{9.36912} \\
\tfrac{1}{2}A &= 53°\ 23'\ 20'' \\
\tfrac{1}{2}B &= 23°\ 26'\ 37'' \\
\tfrac{1}{2}C &= \underline{13°\ 10'\ \ 3''} \\
A &= 106°\ 46'\ 40'' \\
B &= 46°\ 53'\ 14'' \\
C &= \underline{26°\ 20'\ \ 6''} \\
\textit{Check.}\quad A + B + C &= 180°\ \ 0'\ \ 0''
\end{aligned}$$

Even if no mistakes are made in the work, the sum of the three angles found as above may differ very slightly from 180° in consequence of the fact that computation with logarithms is at best only a method of close approximation. When a difference of this kind exists, it should be divided among the angles according to the probable amount of error for each angle.

Exercise 55. Finding the Angles

Find the three angles of a triangle, given the three sides as follows:

1. 51, 65, 20.
2. 78, 101, 29.
3. 111, 145, 40.
4. 21, 26, 31.
5. 19, 34, 49.
6. 43, 50, 57.
7. 37, 58, 79.
8. 73, 82, 91.
9. $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$.
10. 21, 28, 35.
11. 6, 8, 10.
12. 6, 6, 10.
13. 6, 6, 6.
14. 6, 9, 12.
15. 3, 4, 5.

16. Given $a = 14.5$, $b = 55.4$, and $c = 66.9$, find A, B, and C.

17. Given $a = 2$, $b = \sqrt{6}$, and $c = \sqrt{3} - 1$, find A, B, and C.

18. Given $a = 2$, $b = \sqrt{6}$, and $c = \sqrt{3} + 1$, find A, B, and C.

19. The sides of a triangle are 78.9, 65.4, and 97.3 respectively. Find the largest angle.

20. The sides of a triangle are 487.25, 512.33, and 544.37 respectively. Find the smallest angle.

21. Find the angles of a triangle whose sides are $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $\frac{\sqrt{3}-1}{2\sqrt{2}}$, and $\frac{\sqrt{3}}{2}$ respectively.

22. Of three towns, A, B, and C, A is found to be 200 mi. from B and 184 mi. from C, and B is found to be 150 mi. due north from C. How many miles is A north of C?

23. Under what visual angle is an object 7 ft. long seen by an observer whose eye is 5 ft. from one end of the object and 8 ft. from the other end?

24. The sides of a triangle are 14.6 in., 16.7 in., and 18.8 in. respectively. Find the length of the perpendicular from the vertex of the largest angle upon the opposite side.

25. The distances between three cities, A, B, and C, are measured and found to be as follows: $AB = 165$ mi., $AC = 72$ mi., and $BC = 185$ mi. B is due east from A. In what direction is C from A? What two answers are admissible?

26. In a quadrilateral $ABCD$, $AB = 2$ in., $BC = 3$ in., $CD = 3$ in., $DA = 4$ in., and $AC = 4$ in. Find the angles of the quadrilateral.

27. In a parallelogram $ABCD$, $AB = 2$ in., $AC = 3$ in., and $AD = 2.5$ in. Find $\angle CBA$.

28. In a rectangle $ABCD$, $AB = 3.3$ in., and $AC = 5\frac{1}{2}$ in. Find the angles that each diagonal makes with the sides.

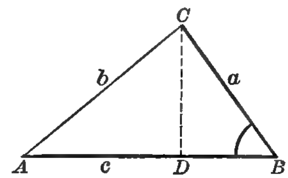
118. Area of a Triangle. The area of a triangle may be found if the following parts are known:

1. Two sides and the included angle;
2. Two angles and any side;
3. The three sides.

These cases will now be considered.

CASE 1. *Given two sides and the included angle.*

Lettering the triangle as here shown, and designating CD by h and the area by S, we have



$$S = \tfrac{1}{2} ch.$$

But $$h = a \sin B.$$

Therefore $$S = \tfrac{1}{2} ac \sin B.$$

Also $S = \frac{1}{2} ab \sin C$, and $S = \frac{1}{2} bc \sin A$.

Exercise 56. Area of a Triangle

Find the areas of the triangles in which it is given that:

1. $a = 27$, $c = 32$, $B = 40°$.
2. $a = 35$, $c = 43$, $B = 37°$.
3. $a = 4.8$, $c = 5.3$, $B = 39°\ 27'$.
4. $a = 9.8$, $c = 7.6$, $B = 48.5°$.
5. $a = 17.3$, $b = 19.4$, $C = 56.25°$.
6. $a = 48.35$, $b = 64.32$, $C = 62°\ 37'$.
7. $b = 127.8$, $c = 168.5$, $A = 72°\ 43'$.
8. $b = 423.9$, $c = 417.8$, $A = 68°\ 27'$.
9. $b = 32.78$, $c = 29.62$, $A = 57°\ 32'\ 20''$.
10. $b = 1487$, $c = 1634$, $A = 61°\ 30'\ 30''$.

11. Prove that the area of a parallelogram is equal to the product of the base, the diagonal, and the sine of the angle included by them.

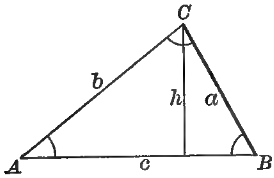
12. Find the area of the quadrilateral $ABCD$, given $AB = 3$ in., $AC = 4.2$ in., $AD = 3.8$ in., $\angle BAD = 88°\ 10'$, $\angle BAC = 36°\ 20'$.

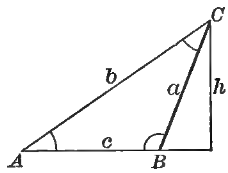
13. In a quadrilateral $ABCD$, $BC = 5.1$ in., $AC = 4.8$ in., $CD = 3.7$ in., $\angle ACB = 123°\ 42'$, and $\angle DCA = 117°\ 26'$. Draw the figure approximately and find the area.

14. In the pentagon $ABCDE$, $AB = 3.1$ in., $AC = 4.2$ in., $AD = 3.7$ in., $AE = 2.9$ in., $\angle A = 132°\ 18'$, $\angle BAC = 38°\ 16'$, and $\angle DAE = 53°\ 9'$. Find the area of the pentagon.

CASE 2. *Given two angles and any side.*

If two angles are known the third can be found, so we may consider that all three angles are given.





Since $$\frac{\sin A}{\sin C} = \frac{a}{c},$$

it follows that $$c = \frac{a \sin C}{\sin A}.$$

And since $$S = \tfrac{1}{2} ac \sin B \text{ (Case 1)},$$

we have $$S = \tfrac{1}{2} a \frac{a \sin C}{\sin A} \sin B = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

Since all three angles are known we may use this formula; or, since $\sin (B + C) = \sin (180° - A) = \sin A$, we may write it as follows:

$$\boldsymbol{S = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}}.$$

Exercise 57. Area of a Triangle

Find the areas of the triangles in which it is given that:

1. $a = 17$, $B = 48°$, $C = 52°$.
2. $a = 182$, $B = 63.5°$, $C = 78.4°$.
3. $a = 298$, $B = 78.8°$, $C = 95.5°$.
4. $a = 19.8$, $B = 39° 20'$, $C = 88° 40'$.
5. $a = 2487$, $B = 87° 28'$, $C = 69° 32'$.
6. $b = 483.7$, $A = 84° 32'$, $C = 78° 49'$.
7. $b = 527.4$, $A = 73° 42'$, $C = 63° 37'$.
8. $c = 296.3$, $A = 58° 35'$, $B = 42° 36'$.
9. $c = 17.48$, $A = 36° 27' 30''$, $B = 73° 50'$.
10. $c = 96.37$, $A = 42° 23' 35''$, $B = 69° 52' 50''$.

11. In a parallelogram $ABCD$ the diagonal AC makes with the sides the angles $27° 10'$ and $32° 43'$ respectively. AB is 2.8 in. long. What is the area of the parallelogram?

CASE 3. *Given the three sides.*

Since, by § 101, $\sin B = 2 \sin \frac{1}{2} B \cos \frac{1}{2} B,$

and, by § 115, $\sin \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}},$

and $\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}},$

by substituting these values for $\sin \frac{1}{2} B$ and $\cos \frac{1}{2} B$ in the above equation, we have

$$\sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)}.$$

By putting this value for $\sin B$ in the formula of Case 1, we have the following important formula for the area of a triangle:

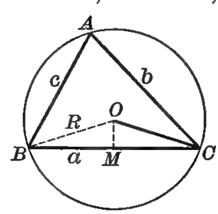
$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

This is known as Heron's Formula for the area of a triangle, having been given in the works of this Greek writer. It is often given in geometry, but the proof by trigonometry is much simpler.

A special case of finding the area of a triangle when the three sides are given is that in which the radius of the circumscribed circle or the radius of the inscribed circle is also given.

If R denotes the radius of the circumscribed circle, we have, from § 106,

$$\sin B = \frac{b}{2R}.$$



By putting this value of $\sin B$ in the formula of Case 1, we have

$$S = \frac{abc}{4R}.$$

If r denotes the radius of the inscribed circle, we may divide the triangle into three triangles by lines from the center of this circle to the vertices; then the altitude of each of the three triangles is equal to r. Therefore

$$S = \tfrac{1}{2} r(a+b+c) = rs.$$

By putting in this formula the value of S from Heron's Formula, we have

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

From this formula, r, as given in § 116, is seen to be equal to the radius of the inscribed circle.

Exercise 58. Area of a Triangle

Find the areas of the triangles in which it is given that:

1. $a = 3,\quad b = 4,\quad c = 5.$
2. $a = 15,\quad b = 20,\quad c = 25.$
3. $a = 10,\quad b = 10,\quad c = 10.$
4. $a = 1.8,\quad b = 3.7,\quad c = 2.1.$
5. $a = 5.3,\quad b = 4.8,\quad c = 4.6.$
6. $a = 7.1,\quad b = 5.3,\quad c = 6.4.$

7. There is a triangular piece of land with sides 48.5 rd., 52.3 rd., and 61.4 rd. Find the area in square rods; in acres.

Find the areas of the triangles in which it is given that:

8. $a = 2.4,\quad b = 3.2,\quad c = 4,\quad R = 2.$
9. $a = 2.7,\quad b = 3.6,\quad c = 4.5,\quad R = 2.25.$
10. $a = 3.9,\quad b = 5.2,\quad c = 6.5,\quad R = 3.25.$
11. $a = 12,\quad b = 12,\quad c = 12,\quad R = 6.928.$

12. Given $a = 60$, $B = 40° 35' 12''$, area $= 12$, find the radius of the inscribed circle.

Find the areas of the triangles in which it is given that:

13. $a = 40,\quad b = 13,\quad c = 37.$
14. $a = 408,\quad b = 41,\quad c = 401.$
15. $a = 624,\quad b = 205,\quad c = 445.$
16. $b = 8,\quad c = 5,\quad A = 60°.$
17. $a = 7,\quad c = 3,\quad A = 60°.$
18. $b = 21.66,\quad c = 36.94,\quad A = 66° 4' 19''.$
19. $a = 215.9,\quad c = 307.7,\quad A = 25° 9' 31''.$
20. $b = 149,\quad A = 70° 42' 30'',\quad B = 39° 18' 28''.$
21. $a = 4474.5,\quad b = 2164.5,\quad C = 116° 30' 20''.$
22. $a = 510,\quad c = 173,\quad B = 162° 30' 28''.$

23. If a is the side of an equilateral triangle, show that the area is $\frac{1}{4} a^2 \sqrt{3}$.

24. Two sides of a triangle are 12.38 ch. and 6.78 ch., and the included angle is $46° 24'$. Find the area.

25. Two sides of a triangle are 18.37 ch. and 13.44 ch., and they form a right angle. Find the area.

26. Two angles of a triangle are $76° 54'$ and $57° 33' 12''$, and the included side is 9 ch. Find the area.

27. The three sides of a triangle are 49 ch., 50.25 ch., and 25.69 ch. Find the area.

28. The three sides of a triangle are 10.64 ch., 12.28 ch., and 9 ch. Find the area.

29. The sides of a triangular field, of which the area is 14 A., are proportional to 3, 5, 7. Find the sides.

30. Two sides of a triangle are 19.74 ch. and 17.34 ch. The first bears N. 82° 30′ W.; the second S. 24° 15′ E. Find the area.

31. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

32. Two sides and the included angle of a triangle are 2416 ft., 1712 ft., and 30°; and two sides and the included angle of another triangle are 1948 ft., 2848 ft., and 150°. Find the sum of their areas.

33. Two adjacent sides of a rectangle are 52.25 ch. and 38.24 ch. Find the area.

34. Two adjacent sides of a parallelogram are 59.8 ch. and 37.05 ch., and the included angle is 72° 10′. Find the area.

35. Two adjacent sides of a parallelogram are 15.36 ch. and 11.46 ch., and the included angle is 47° 30′. Find the area.

36. Show that the area of a quadrilateral is equal to one half the product of its diagonals into the sine of the included angle.

37. The diagonals of a quadrilateral are 34 ft. and 56 ft., intersecting at an angle of 67°. Find the area.

38. The diagonals of a quadrilateral are 75 ft. and 49 ft., intersecting at an angle of 42°. Find the area.

39. In the quadrilateral $ABCD$ we have AB, 17.22 ch.; AD, 7.45 ch.; CD, 14.10 ch.; BC, 5.25 ch.; and the diagonal AC, 15.04 ch. Find the area.

40. Show that the area of a regular polygon of n sides, of which one side is a, is $\frac{na^2}{4}\cot\frac{180°}{n}$.

41. One side of a regular pentagon is 25. Find the area.

42. One side of a regular hexagon is 32. Find the area.

43. One side of a regular decagon is 46. Find the area.

44. If r is the radius of a circle, show that the area of the regular circumscribed polygon of n sides is $nr^2\tan\frac{180°}{n}$, and the area of the regular inscribed polygon is $\frac{n}{2}r^2\sin\frac{360°}{n}$.

45. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.

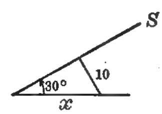
CHAPTER VIII

MISCELLANEOUS APPLICATIONS

119. Applications of the Right Triangle. Although the formulas for oblique triangles apply with equal force to right triangles, yet the formulas developed for the latter in Chapter IV are somewhat simpler and should be used when possible. It will be remembered that these formulas depend merely on the definitions of the functions.

Exercise 59. Right Triangles

1. If the sun's altitude is 30°, find the length of the longest shadow which can be cast on a horizontal plane by a stick 10 ft. in length.



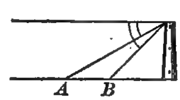
2. A flagstaff 90 ft. high, on a horizontal plane, casts a shadow of 117 ft. Find the altitude of the sun.

3. If the sun's altitude is 60°, what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

4. A tower 93.97 ft. high is situated on the bank of a river. The angle of depression of an object on the opposite bank is 25° 12′. Find the breadth of the river.

5. The angle of elevation of the top of a tower is 48° 19′, and the distance of the base from the point of observation is 95 ft. Find the height of the tower and the distance of its top from the point of observation.

6. From a tower 58 ft. high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are 30° 13′ and 45° 46′. Find the distance between these two objects.



7. From one edge of a ditch 36 ft. wide the angle of elevation of the top of a wall on the opposite edge is 62° 39′. Find the length of a ladder that will just reach from the point of observation to the top of the wall.

8. The top of a flagstaff has been partly broken off and touches the ground at a distance of 15 ft. from the foot of the staff. If the length of the broken part is 39 ft., find the length of the whole staff.

9. From a balloon which is directly above one town the angle of depression of another town is observed to be $10° 14'$. The towns being 8 mi. apart, find the height of the balloon.

10. A ladder 40 ft. long reaches a window 33 ft. high, on one side of a street. Being turned over upon its foot, the ladder reaches another window 21 ft. high, on the opposite side of the street. Find the width of the street.

11. From a mountain 1000 ft. high the angle of depression of a ship is $27° 35' 11''$. Find the distance of the ship from the summit of the mountain.

12. From the top of a mountain 3 mi. high the angle of depression of the most distant object which is visible on the earth's surface is found to be $2° 13' 50''$. Find the diameter of the earth.

13. A lighthouse 54 ft. high is situated on a rock. The angle of elevation of the top of the lighthouse, as observed from a ship, is $4° 52'$, and the angle of elevation of the top of the rock is $4° 2'$. Find the height of the rock and its distance from the ship.

14. The latitude of Cambridge, Massachusetts, is $42° 22' 49''$. What is the length of the radius of that parallel of latitude?

15. At what latitude is the circumference of the parallel of latitude equal to half the equator?

16. In a circle with a radius of 6.7 is inscribed a regular polygon of thirteen sides. Find the length of one of its sides.

17. A regular heptagon, one side of which is 5.73, is inscribed in a circle. Find the radius of the circle.

18. When the moon is setting at any place, the angle at the moon subtended by the earth's radius passing through that place is $57' 3''$. If the earth's radius is 3956.2 mi., what is the moon's distance from the earth's center?

19. A man in a balloon observes the angle of depression of an object on the ground, bearing south, to be $35° 30'$; the balloon drifts $2\frac{1}{2}$ mi. east at the same height, when the angle of depression of the same object is $23° 14'$. Find the height of the balloon.

20. The angle at the earth's center subtended by the sun's radius is $16' 2''$, and the sun's distance is 92,400,000 mi. Find the sun's diameter in miles.

21. A man standing south of a tower and on the same horizontal plane observes its angle of elevation to be $54° 16'$; he goes east 100 yd. and then finds its angle of elevation is $50° 8'$. Find the height of the tower.

22. A regular pyramid, with a square base, has a lateral edge 150 ft. long, and the side of the base is 200 ft. Find the inclination of the face of the pyramid to the base.

23. The height of a house subtends a right angle at a window on the other side of the street, and the angle of elevation of the top of the house from the same point is 60°. The street is 30 ft. wide. How high is the house?

24. The perpendicular from the vertex of the right angle of a right triangle divides the hypotenuse into two segments 364.3 ft. and 492.8 ft. in length respectively. Find the acute angles of the triangle.

25. The bisector of the right angle of a right triangle divides the hypotenuse into two segments 431.9 ft. and 523.8 ft. in length respectively. Find the acute angles of the triangle.

26. Find the number of degrees, minutes, and seconds in an arc of a circle, knowing that the chord which subtends it is 238.25 ft., and that the radius is 196.27 ft.

27. Calculate to the nearest hundredth of an inch the chord which subtends an arc of $37° 43'$ in a circle having a radius of 542.35 in.

28. Calculate to the nearest hundredth of an inch the chord which subtends an arc of 14° in a circle having a radius of 475.23 in.

29. In an isosceles triangle ABC the base AB is 1235 in., and $\angle A = \angle B = 64° 22'$. Find the radius of the inscribed circle.

30. Find the number of degrees, minutes, and seconds in an arc of a circle, knowing that the chord which subtends it is two thirds of the diameter.

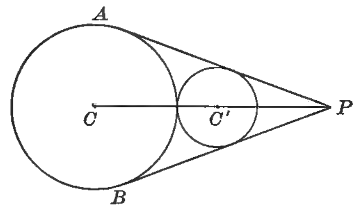
31. Find the number of degrees, minutes, and seconds in an arc of a circle, knowing that the chord which subtends it is three fourths of the diameter.

32. The radius of a circle being 2548.36 in., and the length of a chord BC being 3609.02 in., find the angle BAC made by two tangents drawn at B and C respectively.

33. Find the ratio of a chord to the diameter, knowing that the chord subtends an arc $27° 48'$. If the diameter is 8 in., how long is the chord? If the chord is 8 in., how long is the diameter?

34. Find the length of the diameter of a regular pentagon of which the side is 1 in., and the length of the side of a regular pentagon of which the diameter is 1 in.

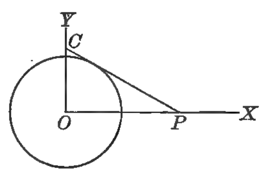
35. Two circles of radii a and b are externally tangent. The common tangents AP, BP, and the line of centers $CC'P$ are drawn as shown in the figure. Find sin APC.



36. In Ex. 35 find $\angle APC$, knowing that $a = 3b$.

37. In $\triangle ABC$, $\angle A = 68° 26' 27''$, $\angle B = 75° 8' 23''$, and the altitude h, from C, is 148.17 in. Required the lengths of the three sides.

38. Two axes, OX and OY, form a right angle at O, the center of a circle of radius 1091 ft. Through P, a point on OX 1997 ft. from O, a tangent is drawn, meeting OY at C. Required OC and the angle CPO.



39. Find the sine of the angle formed by the intersection of the diagonals of a cube.

40. The angle of elevation of the top of a tower observed at a place A, south of it, is 30°; and at a place B, west of A, and at a distance of a from it, the angle of elevation is 18°. Show that the height of the tower is $\dfrac{a}{\sqrt{2+2\sqrt{5}}}$, the tangent of 18° being $\dfrac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$.

41. Standing directly in front of one corner of a flat-roofed house, which is 150 ft. in length, I observe that the horizontal angle which the length subtends has for its cosine $\sqrt{\frac{1}{5}}$, and that the vertical angle subtended by its height has for its sine $\dfrac{3}{\sqrt{34}}$. What is the height of the house?

42. At a distance a from the foot of a tower, the angle of elevation A of the top of the tower is the complement of the angle of elevation of a flagstaff on top of it. Show that the length of the staff is $2a \cot 2A$.

43. A rectangular solid is 4 in. long, 3 in. wide, and 2 in. high. Calculate the tangent of the angle formed by the intersection of any two of the diagonals.

44. Calculate the tangent as in Ex. 43, the solid being l units long, w units wide, and h units high.

120. Applications of the Oblique Triangle. As stated in § 119, when conditions permit of using a right triangle in making a trigonometric observation it is better to do so. Often, however, it is impossible or inconvenient to use the right triangle, as in the case of an observation on an inclined plane, and in such cases resort to the oblique triangle is necessary.

Exercise 60. Oblique Triangles

1. Show how to determine the height of an inaccessible object situated on a horizontal plane by observing its angles of elevation at two points in the same line with its base and measuring the distance between these two points.

2. Show how to determine the height of an inaccessible object standing on an inclined plane.

3. Show how to determine the distance between two inaccessible objects by observing angles at the ends of a line of known length.

4. The angle of elevation of the top of an inaccessible tower standing on a horizontal plain is $63° 26'$; at a point 500 ft. farther from the base of the tower the angle of elevation of the top is $32° 14'$. Find the height of the tower.

5. A tower stands on the bank of a river. From the opposite bank the angle of elevation of the top of the tower is $60° 13'$, and from a point 40 ft. further off the angle of elevation is $50° 19'$. Find the width of the river.

6. At the distance of 40 ft. from the foot of a vertical tower on an inclined plane, the tower subtends an angle of $41° 19'$; at a point 60 ft. farther away the angle subtended by the tower is $23° 45'$. Find the height of the tower.

7. A building makes an angle of $113° 12'$ with the inclined plane on which it stands; at a distance of 89 ft. from its base, measured down the plane, the angle subtended by the building is $23° 27'$. Find the height of the building.

8. A person goes 70 yd. up a slope of 1 in $3\frac{1}{2}$ from the bank of a river and observes the angle of depression of an object on the opposite bank to be $2\frac{1}{4}°$. Find the width of the river.

9. A tree stands on a declivity inclined $15°$ to the horizon. A man ascends the declivity 80 ft. from the foot of the tree and finds the angle then subtended by the tree to be $30°$. Find the height of the tree.

10. The angle subtended by a tree on an inclined plane is, at a certain point, $42° 17'$, and 325 ft. further down it is $21° 47'$. The inclination of the plane is $8° 53'$. Find the height of the tree.

11. From a point B at the foot of a mountain, the angle of elevation of the top A is $60°$. After ascending the mountain one mile, at an inclination of $30°$ to the horizon, and reaching a point C, an observer finds that the angle ACB is $135°$. Find the number of feet in the height of the mountain.

12. The length of a lake subtends, at a certain point, an angle of $46° 24'$, and the distances from this point to the two ends of the lake are 346 ft. and 290 ft. Find the length of the lake.

13. Along the bank of a river is drawn a base line of 500 ft. The angular distance of one end of this line from an object on the opposite side of the river, as observed from the other end of the line, is $53°$; that of the second extremity from the same object, observed at the first, is $79° 12'$. Find the width of the river.

14. Two observers, stationed on opposite sides of a cloud, observe its angles of elevation to be $44° 56'$ and $36° 4'$. Their distance from each other is 700 ft. What is the height of the cloud?

15. From the top of a house 42 ft. high the angle of elevation of the top of a pole is $14° 13'$; at the bottom of the house it is $23° 19'$. Find the height of the pole.

16. From a window on a level with the bottom of a steeple the angle of elevation of the top of the steeple is $40°$, and from a second window 18 ft. higher the angle of elevation is $37° 30'$. Find the height of the steeple.

17. The sides of a triangle are 17, 21, 28. Prove that the length of a line bisecting the longest side and drawn from the opposite angle is 13.

18. The sum of the sides of a triangle is 100. The angle at A is double that at B, and the angle at B is double that at C. Determine the sides.

19. A ship sailing north sees two lighthouses 8 mi. apart in a line due west; after an hour's sailing, one lighthouse bears S.W., and the other S. $22° 30'$ W. ($22° 30'$ west of south). Find the ship's rate.

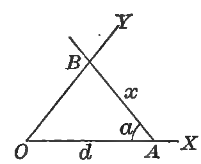
20. A ship, 10 mi. S.W. of a harbor, sees another ship sail from the harbor in a direction S. $80°$ E., at a rate of 9 mi. an hour. In what direction and at what rate must the first ship sail in order to catch up with the second ship in $1\frac{1}{2}$ hr.?

21. Two ships are a mile apart. The angular distance of the first ship from a lighthouse on shore, as observed from the second ship, is 35° 14′ 10″; the angular distance of the second ship from the lighthouse, observed from the first ship, is 42° 11′ 53″. Find the distance in feet from each ship to the lighthouse.

22. A lighthouse bears N. 11° 15′ E., as seen from a ship. The ship sails northwest 30 mi., and then the lighthouse bears east. How far is the lighthouse from the second point of observation?

23. Two rocks are seen in the same straight line with a ship, bearing N. 15° E. After the ship has sailed N.W. 5 mi., the first rock bears E., and the second N.E. Find the distance between the rocks.

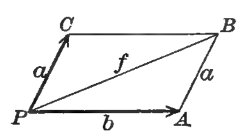
24. On the side OX of a given angle XOY a point A is taken such that $OA = d$. Deduce a formula for the length AB of a line from A to OY that makes a given angle a with OX. From this formula, x is a minimum when what sine is the maximum? Under those circumstances what is the sum of O and a? Then what is the size of $\angle B$? State the conclusion as to the size of $\angle a$ in order that x shall be the minimum.



25. Three points, A, B, and C, form the vertices of an equilateral triangle, AB being 500 ft. Each of the two sides AB and AC is seen from a point P under an angle of 120°; that is, $\angle APB = 120° = \angle CPA$. Find the length of AP.

26. A lighthouse facing south sends out its rays extending in a quadrant from S.E. to S.W. A steamer sailing due east first sees the light when 6 mi. away from the lighthouse and continues to see it for 45 min. At what rate is the ship sailing?

27. If two forces, represented in intensity by the lengths a and b, pull from P in the directions PC and PA, respectively, and if $\angle APC$ is known, the resultant force is represented in intensity and direction by f, the diagonal of the parallelogram $ABCP$. Show how to find f and $\angle APB$, given a, b, and $\angle APC$.



28. Two forces, one of 410 lb. and the other of 320 lb., make an angle of 51° 37′. Find the intensity and the direction of their resultant.

29. An unknown force combined with one of 128 lb. produces a resultant of 200 lb., and this resultant makes an angle of 18° 24′ with the known force. Find the intensity and direction of the unknown force.

30. Wishing to determine the distance between a church A and a tower B, on the opposite side of a river, a man measured a line CD along the river (C being nearly opposite A), and observed the angles ACB, 58° 20′; ACD, 95° 20′; ADB, 53° 30′; BDC, 98° 45′. CD is 600 ft. What is the distance required?

31. Wishing to find the height of a summit A, a man measured a horizontal base line CD, 440 yd. At C the angle of elevation of A is 37° 18′, and the horizontal angle between D and the summit of the mountain is 76° 18′; at D the horizontal angle between C and the summit is 67° 14′. Find the height.

32. A balloon is observed from two stations 3000 ft. apart. At the first station the horizontal angle of the balloon and the other station is 75° 25′, and the angle of elevation of the balloon is 18°. The horizontal angle of the first station and the balloon, measured at the second station, is 64° 30′. Find the height of the balloon.

33. At two stations the height of a kite subtends the same angle A. The angle which the line joining one station and the kite subtends at the other station is B; and the distance between the two stations is a. Show that the height of the kite is $\frac{1}{2} a \sin A \sec B$.

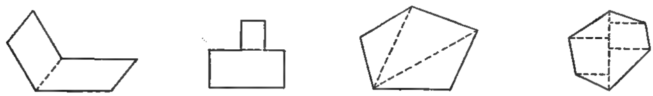
34. Two towers on a horizontal plain are 120 ft. apart. A person standing successively at their bases observes that the angle of elevation of one is double that of the other; but when he is halfway between the towers, the angles of elevation are complementary. Prove that the heights of the towers are 90 ft. and 40 ft.

35. To find the distance of an inaccessible point C from either of two points A and B, having no instruments to measure angles. Prolong CA to a, and CB to b, and draw AB, Ab, and Ba. Measure AB, 500 ft.; aA, 100 ft.; aB, 560 ft.; bB, 100 ft.; and Ab, 550 ft. Compute the distances AC and BC.

36. To compute the horizontal distance between two inaccessible points A and B when no point can be found whence both can be seen. Take two points C and D, distant 200 yd., so that A can be seen from C, and B from D. From C measure CF, 200 yd. to F, whence A can be seen; and from D measure DE, 200 yd. to E, whence B can be seen. Measure AFC, 83°; ACD, 53° 30′; ACF, 54° 31′; BDE, 54° 30′; BDC, 156° 25′; DEB, 88° 30′. Compute the distance AB.

37. A column in the north temperate zone is S. 67° 30′ E. of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 ft. in length, and the elevation of the column at the observer's station is 45°. Find the height of the column.

121. Areas. In finding the areas of rectilinear figures the effort is made to divide any given figure into rectangles, parallelograms, triangles, or trapezoids, unless it already has one of these forms.



For example, the dotted lines show how the above figures may be divided for the purpose of computing the areas. A regular polygon would be conveniently divided into congruent isosceles triangles by the radii of the circumscribed circle.

Exercise 61. Miscellaneous Applications

1. In the trapezoid $ABCD$ it is known that $\angle A = 90°$, $\angle B = 32° 25'$, $AB = 324.35$ ft., and $CD = 208.15$ ft. Find the area.

2. Find the area of a regular pentagon of which each side is 4 in.

3. Find the area of a regular hexagon of which each side is 4 in.

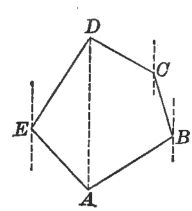
4. The area of a regular polygon inscribed in a circle is to that of the circumscribed regular polygon of the same number of sides as 3 to 4. Find the number of sides.

5. The area of a regular polygon inscribed in a circle is the geometric mean between the areas of the inscribed and circumscribed regular polygons of half the number of sides.

6. Find the ratio of a square inscribed in a circle to a square circumscribed about the same circle. Find the ratio of the perimeters.

7. The square circumscribed about a circle is four thirds the inscribed regular dodecagon.

8. In finding the area of a field $ABCDE$ a surveyor measured the lengths of the sides and the angle which each side makes with the meridian (north and south) line through its extremities. AD happened to be a meridian line. Show how he could compute the area.



9. Two sides of a triangle are 3 and 12, and the included angle is 30°. Find the hypotenuse of the isosceles right triangle of equal area.

10. In the quadrilateral $ABCD$ we have given AB, BC, $\angle A$, $\angle B$, and $\angle C$. Show how to find the area of the quadrilateral.

11. In Ex. 10, suppose $AB = 175$ ft., $BC = 198$ ft., $\angle A = 95°$, $\angle B = 92° 15'$, and $\angle C = 96° 45'$. What is the area?

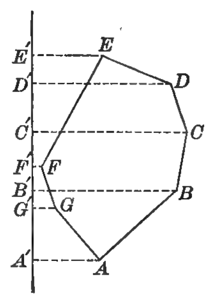
122. Surveyor's Measures. In measuring city lots surveyors commonly use feet and square feet, with decimal parts of these units. In measuring larger pieces of land the following measures are used:

$$16\tfrac{1}{2} \text{ feet (ft.)} = 1 \text{ rod (rd.)}$$
$$66 \text{ feet} = 4 \text{ rods} = 1 \text{ chain (ch.)}$$
$$100 \text{ links (li.)} = 1 \text{ chain}$$
$$10 \text{ square chains (sq. ch.)} = 160 \text{ square rods (sq. rd.)} = 1 \text{ acre (A.)}$$

We may write either 7 ch. 42 li. or 7.42 ch. for 7 chains and 42 links. The decimal fraction is rapidly replacing the old plan, in which the word *link* was used. Similarly, the parts of an acre are now written in the decimal form instead of, as formerly, in square chains or square rods.

Areas are computed as if the land were flat, or projected on a horizontal plane, no allowance being made for inequalities of surface.

123. Area of a Field. The areas of fields are found in various ways, depending upon the shape. In general, however, the work is reduced to the finding of the areas of triangles or trapezoids.



For example, in the case here shown we may draw a north and south line $E'A'$ and then find the sum of the areas of the trapezoids $ABB'A'$, $BCC'B'$, $CDD'C'$, and $DEE'D'$. From this we may subtract the sum of the trapezoids $AGG'A'$, $GFF'G'$ and $FEE'F'$. The result will be the area of the field.

Instead of running the imaginary line $E'A'$ outside the field, it would be quite as convenient to let it pass through F, A, E, or C. The method of computing the area is substantially the same in both cases.

For details concerning surveying, beyond what is here given and is included in Exercise 60, the student is referred to works upon the subject.

Exercise 62. Area of a Field

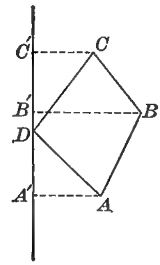
1. Find the number of acres in a triangular field of which the sides are 14 ch., 16 ch., and 20 ch.

2. Find the number of acres in a triangular field having two sides 16 ch. and 30 ch., and the included angle 64° 15′.

3. Find the number of acres in a triangular field having two angles 68.4° and 47.2°, and the included side 20 ch.

4. Required the area of the field described in § 123, knowing that $AA' = 8$ ch., $BB' = 12$ ch., $CC' = 13$ ch., $DD' = 12$ ch., $EE' = 8$ ch., $FF' = 1$ ch., $GG' = 2$ ch., $A'G' = 6$ ch., $G'B' = 1.5$ ch., $B'F' = 2.3$ ch., $F'C' = 3$ ch., $C'D' = 4$ ch., $D'E' = 2.9$ ch.

5. In a quadrangular field $ABCD$, AB runs N. 27° E. 12.5 ch., BC runs N. 30° W. 10 ch., CD runs S. 37° W. 15 ch., and DA runs S. 45° E. 12 ch. Find the area.



That AB is N. 27° E. means that it makes an angle of 27° east of the line running north through A.

6. In a triangular field ABC, AB runs N. 10° E. 30 ch., BC runs S. 30° W. 19 ch., and CA runs S. 30° E. 16 ch. Find the area.

7. In a field $ABCD$, AB runs E. 10 ch., BC runs N. 12 ch., CD runs S. 68° 12′ W. 10.77 ch., and DA runs S. 8 ch. Find the area.

8. A field is in the form of a right triangle of which the sides are 15 ch., 20 ch., and 25 ch. From the vertex of the right angle a line is run to the hypotenuse, making an angle of 30° with the side that is 15 ch. long. Find the area of each of the triangles into which the field is divided.

Using a protractor, draw to scale the fields referred to in the following examples, and find the areas:

9.	AB, N. 72° E. 18 ch.,	CD, N. 68° W. 21 ch.,
	BC, N. 10° E. 12.5 ch.,	DA, S. 12° E. 26.3 ch.
10.	AB, N. 45° E. 10 ch.,	CD, S. 15° W. 18.21 ch.,
	BC, S. 75° E. 11.55 ch.,	DA, N. 45° W. 19.11 ch.
11.	AB, N. 5° 30′ W. 6.08 ch.,	CD, S. 3° E. 5.33 ch.,
	BC, S. 82° 30′ W. 6.51 ch.,	DA, E. 6.72 ch.
12.	AB, N. 6° 15′ W. 6.31 ch.,	CD, S. 5° E. 5.86 ch.,
	BC, S. 81° 50′ W. 4.06 ch.,	DA, N. 88° 30′ E. 4.12 ch.

13. A farm is bounded and described as follows: Beginning at the southwest corner of lot No. 13, thence N. $1\frac{1}{4}$° E. 132 rods and 23 links to a stake in the west boundary line of said lot; thence S. 89° E. 32 rods and $15\frac{4}{10}$ links to a stake; thence N. $1\frac{1}{4}$° E. 29 rods and 15 links to a stake in the north boundary line of said lot; thence S. 89° E. 61 rods and $18\frac{6}{10}$ links to a stake; thence S. $32\frac{1}{2}$° W. 54 rods to a stake; thence S. $35\frac{1}{4}$° E. 22 rods and 4 links to a stake; thence S. 48° E. 33 rods and 2 links to a stake; thence S. $7\frac{1}{2}$° W. 76 rods and 20 links to a stake in the south boundary line of said lot; thence N. 89° W. 96 rods and 10 links to the place of beginning. Containing 85.65 acres, more or less. Verify the area given and plot the farm.

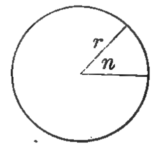
This is a common way of describing a farm in a deed or a mortgage.

124. The Circle. It is learned in geometry that

$$c = 2\pi r, \quad \text{and} \quad a = \pi r^2,$$

where c = circumference, r = radius, a = area, and $\pi = 3.14159+ = 3.1416- =$ about $3\frac{1}{7}$. For practical purposes $\frac{22}{7}$ may be taken.

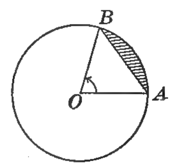
Furthermore, if we have a sector with angle n degrees, the area of the sector is evidently $\dfrac{n}{360}$ of πr^2.



From these formulas we can, by the help of the formulas relating to triangles, solve numerous problems relating to the circle.

Exercise 63. The Circle

1. A sector of a circle of radius 8 in. has an angle of 62.5°. A chord joining the extremities of the radii forming the sector cuts off a segment. What is the area of this segment?

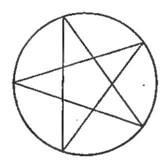


2. A sector of a circle of diameter 9.2 in. has an angle of 29° 42′. A chord joining the extremities of the radii forming the sector cuts off a segment. What is the area of the remainder of the circle?

3. In a circle of radius 3.5 in., what is the area included between two parallel chords of 6 in. and 5 in. respectively? (Give two answers.)

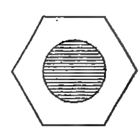
4. A regular hexagon is inscribed in a circle of radius 4 in. What is the area of that part of the circle not covered by the hexagon?

5. In a circle of radius 10 in. a regular five-pointed star is inscribed. What is the area of the star? What is the area of that part of the circle not covered by the star?



6. In a circle of diameter 7.2 in. a regular five-pointed star is inscribed. The points are joined, thus forming a regular pentagon. There is also a regular pentagon formed in the center by the crossing of the lines of the star. The small pentagon is what fractional part of the large one?

7. A circular hole is cut in a regular hexagonal plate of side 8 in. The radius of the circle is 4 in. What is the area of the rest of the plate?



8. A regular hexagon is formed by joining the mid-points of the sides of a regular hexagon. Find the ratio of the smaller hexagon to the larger.

CHAPTER IX

PLANE SAILING

125. Plane Sailing. A simple and interesting application of plane trigonometry is found in that branch of navigation in which the surface of the earth is considered a plane. This can be the case only when the distance is so small that the curvature of the earth may be neglected.

This chapter may be omitted if further applications of a practical nature are not needed.

126. Latitude and Departure. The *difference of latitude* between two places is the arc of a meridian between the parallels of latitude which pass through those places.

Thus the latitude of Cape Cod is $42° 2' 21''$ N. and the latitude of Cape Hatteras is $35° 15' 14''$ N. The difference of latitude is $6° 47' 7''$.

The *departure* between two meridians is the length of the arc of a parallel of latitude cut off by those meridians, measured in geographic miles.

The *geographic mile*, or *knot*, is the length of $1'$ of the equator. Taking the equator to be 131,385,456 ft., $\frac{1}{60}$ of $\frac{1}{360}$ of this length is 6082.66 ft., and this is generally taken as the standard in the United States. The British Admiralty knot is a little shorter, being 6080 ft. The term "mile" in this chapter refers to the geographic mile, and there are 60 mi. in one degree of a great circle.

Calling the *course* the angle between the track of the ship and the meridian line, as in the case of N. 20° E., it will be evident by drawing a figure that the difference in latitude, expressed in distance, equals the distance sailed multiplied by the cosine of the course. That is

$$\text{diff. of latitude} = \text{distance} \times \cos C.$$

In the same way we can find the departure. This is evidently given by the equation

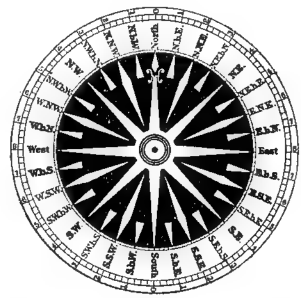
$$\text{departure} = \text{distance} \times \sin C.$$

For example, if a ship has sailed N. 30° E. 10 mi., the difference in latitude, expressed in miles, is

$$10 \cos 30° = 10 \times 0.8660 = 8.66,$$

and the departure is $10 \sin 30° = 10 \times 0.5 = 5.$

127. The Compass. The mariner divides the circle into 32 equal parts called *points*. There are therefore 8 points in a right angle, and a point is 11° 15′. To sail two points east of north means, therefore, to sail 22° 30′ east of north, or north-northeast (N.N.E.) as shown on the compass. Northeast (N.E.) is 45° east of north. One point east of north is called north by east (N. by E.) and one point east of south is called south by east (S. by E.). The other terms used, and their significance in angular measure, will best be understood from the illustration and the following table:



North		Points	° ′ ″	Points	South	
N. by E.	N. by W.	0-¼ 0-½ 0-¾ 1	2 48 45 5 37 30 8 26 15 11 15 0	0-¼ 0-½ 0-¾ 1	S. by E.	S. by W.
N.N.E.	N.N.W.	1-¼ 1-½ 1-¾ 2	14 3 45 16 52 30 19 41 15 22 30 0	1-¼ 1-½ 1-¾ 2	S.S.E.	S.S.W.
N.E. by N.	N.W. by N.	2-¼ 2-½ 2-¾ 3	25 18 45 28 7 30 30 56 15 33 45 0	2-¼ 2-½ 2-¾ 3	S.E. by S.	S.W. by S.
N.E.	N.W.	3-¼ 3-½ 3-¾ 4	36 33 45 39 22 30 42 11 15 45 0 0	3-¼ 3-½ 3-¾ 4	S.E.	S.W.
N.E. by E.	N.W. by W.	4-¼ 4-½ 4-¾ 5	47 48 45 50 37 30 53 26 15 56 15 0	4-¼ 4-½ 4-¾ 5	S.E. by E.	S.W. by W.
E.N.E.	W.N.W.	5-¼ 5-½ 5-¾ 6	59 3 45 61 52 30 64 41 15 67 30 0	5-¼ 5-½ 5-¾ 6	E.S.E.	W.S.W.
E. by N.	W. by N.	6-¼ 6-½ 6-¾ 7	70 18 45 73 7 30 75 56 15 78 45 0	6-¼ 6-½ 6-¾ 7	E. by S.	W. by S.
E.	W.	7-¼ 7-½ 7-¾ 8	81 33 45 84 22 30 87 11 15 90 0 0	7-¼ 7-½ 7-¾ 8	E.	W.

The compass varies in different parts of the earth; hence, in sailing, the compass course is not the same as the true course. The true course is the compass course, with allowances for variation of the needle in different parts of the earth, for deviation caused by the iron in the ship, and for leeway, the angle which the ship makes with her track.

Exercise 64. Plane Sailing

1. A ship sails from latitude 40° N. on a course N.E. 26 mi. Find the difference of latitude and the departure.

2. A ship sails from latitude 35° N. on a course S.W. 53 mi. Find the difference of latitude and the departure.

3. A ship sails from a point on the equator on a course N.E. by N. 62 mi. Find the difference of latitude and the departure.

4. A ship sails from latitude 43° 45′ S. on a course N. by E. 38 mi. Find the difference of latitude and the departure.

5. A ship sails from latitude 1° 45′ N. on a course S.E. by E. 25 mi. Find the difference of latitude and the departure.

6. A ship sails from latitude 13° 17′ S. on a course N.E. by E. $\frac{3}{4}$ E., until the departure is 42 mi. Find the difference of latitude and the latitude reached.

7. A ship sails from latitude 40° 20′ N. on a N.N.E. course for 92 mi. Find the departure.

8. If a steamer sails S.W. by W. 20 mi. what is the departure and the difference of latitude?

9. If a sailboat sails N. 25° W. until the departure is 25 mi., what distance does it sail?

10. A ship sails from latitude 37° 40′ N. on a N.E. by E. course for 122 mi. Find the departure.

11. A yacht sails $6\frac{1}{2}$ points west of north, the distance being 12 mi. What is the departure?

12. A steamer sails S.W. by W. 28 mi. It then sails N.W. 30 mi. How far is it then to the west of its starting point?

13. A ship sails on a course between S. and E. 24 mi., leaving latitude 2° 52′ S. and reaching latitude 2° 58′ S. Find the course and the departure.

14. A ship sails from latitude 32° 18′ N., on a course between N. and W., a distance of 34 mi. and a departure of 10 mi. Find the course and the latitude reached.

15. A ship sails on a course between S. and E., making a difference of latitude 13 mi. and a departure of 20 mi. Find the distance and the course.

16. A ship sails on a course between N. and W., making a difference of latitude 17 mi. and a departure of 22 mi. Find the distance and the course.

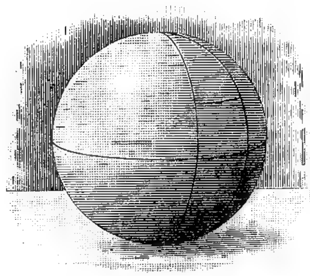
128. Parallel Sailing. Sailing due east or due west, remaining on the same parallel of latitude, is called *parallel sailing.*

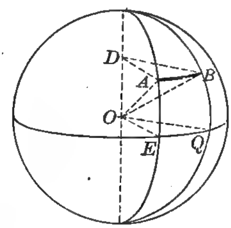
129. Finding Difference in Longitude. In parallel sailing the distance sailed is, by definition (§ 126), the departure. From the departure the difference in longitude is found as follows:

Let AB be the departure. Then in rt. $\triangle\, OAD$

$$\angle AOD = 90° - \text{lat.}$$

Hence $$\frac{DA}{OA} = \sin(90° - \text{lat.}) = \cos \text{lat.}$$





The triangles DAB and OEQ are similar, the arcs being (§ 125) considered straight lines.

Therefore $$\frac{DA}{OE} = \frac{AB}{EQ}, \quad \text{or} \quad \frac{DA}{OA} = \frac{AB}{EQ}.$$

Hence $$\cos \text{lat.} = \frac{AB}{EQ}.$$

Therefore $$EQ = \frac{AB}{\cos \text{lat.}} = AB \times \sec \text{lat.}$$

That is, **Diff. long. = depart. × sec lat.**

That is, the number of minutes in the difference in longitude is the product of the number of miles in the departure by the secant of the latitude, the nautical, or geographic, mile being a minute of longitude on the equator.

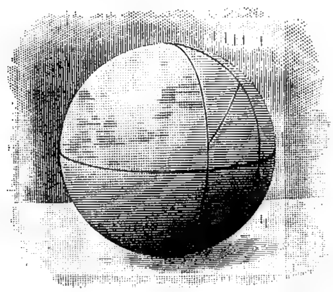
Exercise 65. Parallel Sailing

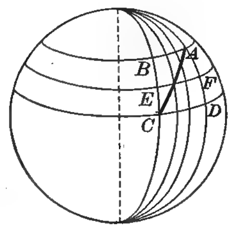
1. A ship in latitude 42° 16′ N., longitude 72° 16′ W., sails due east a distance of 149 mi. What is the position of the point reached?

2. A ship in latitude 44° 49′ S., longitude 119° 42′ E., sails due west until it reaches longitude 117° 16′ E. Find the distance made.

3. A ship in latitude 60° 15′ N., longitude 60° 15′ W., sails due west a distance of 60 mi. What is the position of the point reached?

130. Middle Latitude Sailing. Since a ship rarely sails for any length of time due east or due west, the difference in longitude cannot ordinarily be found as in parallel sailing (§§ 128, 129). Therefore, in plane sailing the departure between two places is measured generally on that parallel of latitude which lies midway between the





parallels of the two places. This is called the method of *middle latitude sailing.* Hence, in middle latitude sailing,

Diff. long. = depart. × sec mid. lat.

This assumption produces no great error, except in very high latitudes or excessive runs.

Exercise 66. Middle Latitude Sailing

1. A ship leaves latitude 31° 14′ N., longitude 42° 19′ W., and sails E.N.E. 32 mi. Find the position reached.

2. Leaving latitude 49° 57′ N., longitude 15° 16′ W., a ship sails between S. and W. till the departure is 38 mi. and the latitude is 49° 38′ N. Find the course, distance, and longitude reached.

3. Leaving latitude 42° 30′ N., longitude 58° 51′ W., a ship sails S.E. by S. 48 mi. Find the position reached.

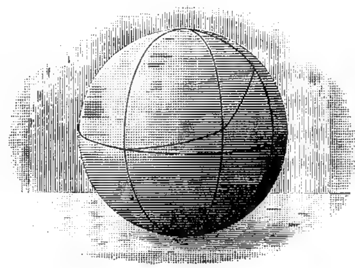
4. Leaving latitude 49° 57′ N., longitude 30° W., a ship sails S. 39° W. and reaches latitude 49° 44′ N. Find the distance and the longitude reached.

5. Leaving latitude 37° N., longitude 32° 16′ W., a ship sails between N. and W. 45 mi. and reaches latitude 37° 10′ N. Find the course and the longitude reached.

6. A ship sails from latitude 40° 28′ N., longitude 74° W., on an E.S.E. course, 62 mi. Find the latitude and longitude reached.

7. A ship sails from latitude 42° 20′ N., longitude 71° 4′ W., on a N.N.E. course, 30 mi. Find the latitude and longitude reached.

131. Traverse Sailing. In case a ship sails from one point to another on two or more different courses, the departure and difference of longitude are found by reckoning each course separately and combining the results. For example, two such courses are shown in the figure. This is called the method of *traverse sailing*.



No new principles are involved in traverse sailing, as will be seen in solving Ex. 1, given below.

Exercise 67. Traverse Sailing

1. Leaving latitude 37° 16′ S., longitude 18° 42′ W., a ship sails N.E. 104 mi., then N.N.W. 60 mi., then W. by S. 216 mi. Find the position reached, and its bearing and distance from the point left.

For the first course we have difference of latitude 73.5 N., departure 73.5 E.; for the second course, difference of latitude 55.4 N., departure 23 W.; for the third course, difference of latitude 42.1 S., departure 211.8 W.

On the whole, then, the ship has made 128.9 mi. of north latitude and 42.1 mi. of south latitude. The place reached is therefore on a parallel of latitude 86.8 mi. to the north of the parallel left; that is, in latitude 35° 49.2′ S.

In the same way the departure is found to be 161.3 mi. W., and the middle latitude is 36° 32.6′. With these data we find the difference of longitude to be 201′, or 3° 21′ W. Hence the longitude reached is 22° 3′ W.

With the difference of latitude 86.8 mi. and the departure 161.3 mi., we find the course to be N. 61° 43′ W. and the distance 183.2 mi. The ship has reached the same point that it would have reached if it had sailed directly on a course N. 61° 43′ W. for a distance of 183.2 mi.

2. A ship leaves Cape Cod (42° 2′ N., 70° 3′ W.) and sails S.E. by S. 114 mi., then N. by E. 94 mi., then W.N.W. 42 mi. Find its position and the total distance.

3. A ship leaves Cape of Good Hope (34° 22′ S., 18° 30′ E.) and sails N.W. 126 mi., then N. by E. 84 mi., then W.S.W. 217 mi. Find its position and the total distance.

4. A ship in latitude 40° N. and longitude 67° 4′ W. sails N.W. 60 mi., then N. by W. 52 mi., then W.S.W. 83 mi. Find its position.

5. A ship sailed S.S.W. 48 mi., then S.W. by S. 36 mi., and then N.E. 24 mi. Find the difference in latitude and the departure.

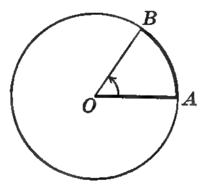
6. A ship sailed S. $\frac{1}{2}$ E. 18 mi., S.W. $\frac{1}{2}$ S. 37 mi., and then S.S.W. $\frac{1}{4}$ W. 56 mi. Find the difference in latitude and the departure.

CHAPTER X

GRAPHS OF FUNCTIONS

132. Circular Measure. Besides the methods of measuring angles which have been discussed already and are generally used in practical work, there is another method that is frequently employed in the theoretical treatment of the subject. This takes for the unit the angle subtended by an arc which is equal in length to the radius, and is known as *circular measure.*

133. Radian. An angle subtended by an arc equal in length to the radius of the circle is called a *radian.*



The term "radian" is applied to both the angle and arc. In the annexed figure we may think of a radius bent around the arc AB so as to coincide with it. Then $\angle AOB$ is a radian.

134. Relation of the Radian to Degree Measure. The number of radians in 360° is equal to the number of times the length of the radius is contained in the length of the circumference. It is proved in geometry that this number is 2π for all circles, π being equal to 3.1416, nearly. Therefore the radian is the same angle in all circles.

The circumference of a circle is 2π times the radius.

Hence $\quad 2\pi$ radians $= 360°$, and π radians $= 180°$.

Therefore $\quad 1 \text{ radian} = \dfrac{180°}{\pi} = 57.29578° = 57°\,17'\,45''$,

and $\quad 1 \text{ degree} = \dfrac{\pi}{180} \text{ radian} = 0.017453 \text{ radian}.$

135. Number of Radians in an Angle. From the definition of radian we see that the number of radians in an angle is equal to the length of the subtending arc divided by the length of the radius.

Thus, if an arc is 6 in. long and the radius of the circle is 4 in., the number of radians in the angle subtended by the arc is 6 in. $\div$ 4 in., or $1\frac{1}{2}$.

This may be reduced to degrees thus:

$$1\tfrac{1}{2} \times 57.29578° = 85.94367°,$$

or, for practical purposes, $\quad 1\tfrac{1}{2} \times 57.3° = 85.9° = 85°\,54'.$

136. Reduction of Radians and Degrees. From the values found in § 134 the following methods of reduction are evident:

To reduce radians to degrees, multiply $57°\ 17'\ 45''$*, or* $57.29578°$*, by the number of radians.*

To reduce degrees to radians, multiply 0.017453 *by the number of degrees.*

These rules need not be learned, since we do not often have to make these reductions. It is essential, however, to know clearly the significance of radian measure, since we shall often use it hereafter. In solving the following problems the rules may be consulted as necessary.

In particular the student should learn the following:

$360° = 2\,\pi$ radians,	$60° = \frac{1}{3}\,\pi$ radians,
$180° = \pi$ radians,	$30° = \frac{1}{6}\,\pi$ radians,
$90° = \frac{1}{2}\,\pi$ radians,	$15° = \frac{1}{12}\,\pi$ radians,
$45° = \frac{1}{4}\,\pi$ radians,	$22.5° = \frac{1}{8}\,\pi$ radians.

The word *radians* is usually understood without being written. Thus $\sin 2\,\pi$ means the sine of $2\,\pi$ radians, or $\sin 360°$; and $\tan \frac{1}{4}\,\pi$ means the tangent of $\frac{1}{4}\,\pi$ radians, or $45°$. Also, $\sin 2$ means the sine of 2 radians, or $\sin 114.59156°$.

Exercise 68. Radians

Express the following in radians:

1. $270°$.	3. $56.25°$.	5. $196.5°$.	7. $200°$.
2. $11.25°$.	4. $7.5°$.	6. $1440°$.	8. $3000°$.

Express the following in degree measure:

9. $1\frac{1}{2}\,\pi$.	11. $1\frac{1}{6}\,\pi$.	13. $\frac{1}{24}\,\pi$.	15. $6\,\pi$.
10. $1\frac{1}{3}\,\pi$.	12. $1\frac{1}{4}\,\pi$.	14. $3\,\pi$.	16. $10\,\pi$.

State the quadrant in which the following angles lie:

17. $\frac{2}{3}\,\pi$.	19. $1\frac{3}{8}\,\pi$.	21. $2.5\,\pi$.	23. 1.
18. $\frac{4}{5}\,\pi$.	20. $1\frac{4}{5}\,\pi$.	22. $-3.4\,\pi$.	24. -2.

Express the following in degrees and also in radians:

25. $\frac{3}{5}$ of four right angles.

26. $\frac{5}{6}$ of four right angles.

27. $\frac{2}{3}$ of two right angles.

28. $\frac{3}{8}$ of one right angle.

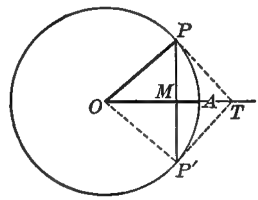
29. What decimal part of a radian is $1°$? $1'$?

30. How many minutes in a radian? How many seconds?

31. Express in radians the angle of an equilateral triangle.

32. Over what part of a radian does the minute hand of a clock move in 15 min.?

137. Functions of Small Angles. Let AOP be any acute angle, and let x be its circular measure. Describe a circle of unit radius about O as center and take $\angle AOP' = -\angle AOP$. Draw the tangents to the circle at P and P', meeting OA in T. Then we see that



$$\text{chord } PP' < \text{arc } PP'$$
$$< PT + P'T.$$

Dividing by 2, $\quad MP < \text{arc } AP < PT,$

or $\quad \sin x < x < \tan x.$

Dividing by $\sin x$, $\quad 1 < \dfrac{x}{\sin x} < \sec x.$

Whence $\quad 1 > \dfrac{\sin x}{x} > \cos x.$

Therefore the value of $\dfrac{\sin x}{x}$ lies between $\cos x$ and 1.

If, now, the angle x is constantly diminished, $\cos x$ approaches the value 1.

Accordingly, the limit of $\dfrac{\sin x}{x}$, as x approaches 0, is 1.

Hence when x denotes the circular measure of an angle near 0° we may use x instead of $\sin x$ and instead of $\tan x$.

For example, required to find the sine and cosine of 1′.

If x is the circular measure of 1′,

$$x = \frac{2\pi}{360 \times 60} = \frac{3.14159+}{10800} = 0.00029088+,$$

the next figure in x being 8.

Now $\sin x > 0$ but $< x$; hence $\sin 1'$ lies between 0 and 0.000290889.

Again, $\cos 1' = \sqrt{1 - \sin^2 1'} > \sqrt{1 - (0.0003)^2} > 0.9999999.$

Hence $\quad \cos 1' = 0.9999999 +.$

But, as above, $\quad \sin x > x \cos x.$

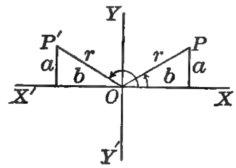
$$\begin{aligned} \therefore \sin 1' &> 0.000290888 \times 0.9999999 \\ &> 0.000290888\,(1 - 0.0000001) \\ &> 0.000290888 - 0.0000000000290888 \\ &> 0.000290887. \end{aligned}$$

Hence $\sin 1'$ lies between 0.000290887 and 0.000290889; that is, to eight places of decimals,

$$\sin 1' = 0.00029088 +,$$

the next figure being 7 or 8.

138. Angles having the Same Sine. If we let $\angle XOP = x$, in this figure, and let P' be symmetric to P with respect to the axis YY', we shall have $\angle XOP' = 180° - x$, or $\pi - x$. And since $\frac{a}{r} = \sin x = \sin(\pi - x)$ we see that x and $\pi - x$ have the same sine.



Furthermore, $\sin x = \sin(360° + x)$, and $\sin(180° - x) = \sin(360° + 180° - x)$. That is, we may increase any angle by 360° without changing the sine. Hence we have $\sin x = \sin(n \cdot 360° + x)$, and $\sin\ (180° - x) = \sin(n \cdot 360° + 180° - x)$. Using circular measure we may write these results as follows:

$$\sin x = \sin(2\,k\pi + x), \text{ and } \sin(\pi - x) = \sin(\overline{2\,k + 1}\,\pi - x).$$

These may be simplified still more, thus:

$$\sin x = \sin[n\pi + (-1)^n x]$$

where n is any integer, positive or negative.

Thus if $n = 0$ we have $\sin x = \sin(0 \cdot \pi + (-1)^0 x) = \sin x$; if $n = 1$ we have $\sin x = \sin(\pi - x)$; if $n = 2$ we have $\sin x = \sin(2\pi + x)$; and so on.

Since the sine is the reciprocal of the cosecant, it is evident that x and $n\pi + (-1)^n x$ have the same cosecant.

To find four angles whose sine is 0.2588, we see by the tables that $\sin 15° = 0.2588$. Hence we have $\sin 15° = \sin[n\pi + (-1)^n \cdot 15°] = \sin(\pi - 15°) = \sin(2\pi + 15°) = \sin(3\pi - 15°)$; and so on.

Exercise 69. Sines and Small Angles

1. Find four angles whose sine is 0.2756.
2. Find six angles whose sine is 0.5000.
3. Find eight angles having the same sine as $\frac{1}{6}\pi$.
4. Find four angles having the same cosecant as $\frac{3}{8}\pi$.
5. Find four angles having the same cosecant as 0.1π.

Given $\pi = 3.141592653589$, compute to eleven decimal places:

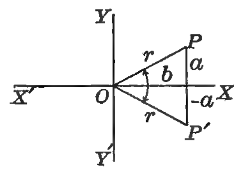
6. $\cos 1'$. 7. $\sin 1'$. 8. $\tan 1'$. 9. $\sin 2'$.

10. From the results of Exs. 6 and 7, and by the aid of the formula $\sin 2x = 2 \sin x \cos x$, compute $\sin 2'$, carrying the multiplication to six decimal places. Compare the result with that of Ex. 9.

11. Compute $\sin 1°$ to four decimal places.

12. From the formula $\cos x = 1 - 2\sin^2\frac{x}{2}$, show that $\cos x > 1 - \frac{x^2}{2}$.

139. Angles having the Same Cosine. If we let $\angle XOP = x$, in this figure, and let P' be symmetric to P with respect to the axis XX', we shall have $\angle XOP' = 360° - x$, or $-x$, depending on whether we think of it as a positive or a negative angle. In either case its cosine is $\frac{b}{r}$, the same as $\cos x$.



In either case $\cos x = \cos(n \cdot 360° - x)$.

In general, $\quad \cos x = \cos(2\,n\pi \pm x)$,

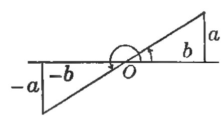
where n is any integer, positive or negative.

Thus if $n = 0$, we have $\cos x = \cos(\pm x)$; if $n = 1$, we have $\cos x = \cos(2\pi \pm x)$; if $n = 2$, we have $\cos x = \cos(4\pi \pm x)$; and so on.

Since the cosine is the reciprocal of the secant, it is evident that x and $2\,n\pi \pm x$ have the same secant.

140. Angles having the Same Tangent. Since we have $\tan x = \frac{a}{b}$, and $\tan(180° + x) = \frac{-a}{-b}$, we see that $\tan x = \tan(180° + x)$. In general we may say that

$$\tan x = \tan(2\,k\pi + x) = \tan(2\,k\pi + \pi + x).$$



This may be written more simply thus:

$$\tan x = \tan(n\pi + x),$$

where n is any integer, positive or negative.

Thus if we have $\tan 20°$ given, we know that $n\pi + 20°$ has the same tangent. Writing both in degree measure, we may say that $n \cdot 180° + 20°$ has the same tangent. If $n = 1$, we have $200°$; if $n = 2$, we have $380°$; if $n = 3$, we have $560°$; and so on. Furthermore, if $n = -1$, we have $-160°$; and so on.

Since the cotangent is the reciprocal of the tangent, it is evident that x and $n\pi + x$ have the same cotangent.

Exercise 70. Angles having the Same Functions

1. Find two positive angles that have $\frac{1}{2}$ as their cosine.
2. Find two negative angles that have $\frac{1}{2}$ as their cosine.
3. Find four angles whose cosine is the same as the cosine of 25°.
4. Find four angles that have 2 as their secant.
5. Find two positive angles that have 1 as their tangent.
6. Find two negative angles that have 1 as their tangent.
7. Find four angles that have $\sqrt{3}$ as their tangent.
8. Find four angles that have $\sqrt{3}$ as their cotangent.
9. Find four angles that have 0.5000 as their tangent.
10. Find four negative angles whose cotangent is 0.5000.

141. Inverse Trigonometric Functions. If $y = \sin x$, then x is the angle whose sine is y. This is expressed by the symbols $x = \sin^{-1} y$, or $x = \text{arc} \sin y$.

In American and English books the symbol $\sin^{-1} y$ is generally used; on the continent of Europe the symbol $\text{arc} \sin y$ is the one that is met.

The symbol $\sin^{-1} y$ is read "the inverse sine of y," "the antisine of y," or "the angle whose sine is y." The symbol $\text{arc} \sin y$ is read "the arc whose sine is y," or "the angle whose sine is y."

The symbols $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, and so on are similarly used.

The symbol $\sin^{-1} y$ must not be confused with $(\sin y)^{-1}$. The former means the angle whose sine is y; the latter means the reciprocal of $\sin y$.

We have seen (§ 138) that $\sin^{-1} 0.5000$ may be 30°, 150°, 390°, 510°, and so on. In other words, there are many values for $\sin^{-1} x$; that is,

Inverse trigonometric functions are many-valued.

142. Principal Value of an Inverse Function. The smallest positive value of an inverse function is called its *principal value.*

For example, the principal value of $\sin^{-1} 0.5000$ is 30°; the principal value of $\cos^{-1} 0.5000$ is 60°; the principal value of $\tan^{-1}(-1)$ is 135°; and so on.

Exercise 71. Inverse Functions

Prove the following formulas:

1. $\sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi$.
2. $\tan^{-1} x + \cot^{-1} x = \frac{1}{2}\pi$.
3. $\sec^{-1} x + \csc^{-1} x = \frac{1}{2}\pi$.
4. $\sin^{-1}(-x) = -\sin^{-1} x$.

Find two values of each of the following:

5. $\sin^{-1} \frac{1}{2}\sqrt{3}$.
6. $\csc^{-1} \sqrt{2}$.
7. $\tan^{-1} \frac{1}{3}\sqrt{3}$.
8. $\tan^{-1} \infty$.
9. $\sec^{-1} 2$.
10. $\cos^{-1}\left(-\frac{1}{2}\sqrt{2}\right)$.

11. Find the value of the sine of the angle whose cosine is $\frac{1}{2}$; that is, the value of $\sin(\cos^{-1}\frac{1}{2})$.

Find the values of the following:

12. $\sin\left(\cos^{-1}\frac{1}{2}\sqrt{3}\right)$.
13. $\sin(\tan^{-1} 1)$.
14. $\cos(\cot^{-1} 1)$.

Prove the following formulas:

15. $\tan(\tan^{-1} x + \tan^{-1} y) = \dfrac{x+y}{1-xy}$.
16. $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \sin^{-1} x$.
17. $\tan(2\tan^{-1} x) = \dfrac{2x}{1-x^2}$.
18. $\sin(2\tan^{-1} x) = \dfrac{2x}{1+x^2}$.

Find four values of each of the following:

19. $\tan^{-1} 0.5774$.
20. $\cot^{-1} 0.6249$.
21. $\sin^{-1} 0.9613$.
22. $\sin^{-1} 0.3256$.
23. $\cot^{-1} 0.2756$.
24. $\cos^{-1} 0.9455$.
25. Solve the equation $y = \sin^{-1} \frac{1}{3}$.
26. Find the value of $\sin(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3})$.
27. If $\sin^{-1} x = 2 \cos^{-1} x$, find the value of x.

Prove the following formulas:

28. $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.
29. $\cos(2 \sin^{-1} x) = 1 - 2x^2$.
30. $\sin(\sin^{-1} x) = x$.
31. $\sin(\sin^{-1} x + \sin^{-1} y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$.
32. $\tan^{-1} 2 + \tan^{-1} \frac{1}{2} = \frac{1}{2}\pi$.
33. $2 \tan^{-1} x = \tan^{-1}[2x : (1 - x^2)]$.
34. $2 \sin^{-1} x = \sin^{-1}(2x\sqrt{1 - x^2})$.
35. $2 \cos^{-1} x = \cos^{-1}(2x^2 - 1)$.
36. $3 \tan^{-1} x = \tan^{-1}[(3x - x^3) : (1 - 3x^2)]$.
37. $\sin^{-1}\sqrt{x : y} = \tan^{-1}\sqrt{x : (y - x)}$.
38. $\sin^{-1}\sqrt{(x - y) : (x - z)} = \tan^{-1}\sqrt{(x - y) : (y - z)}$.
39. $\sin^{-1} x = \sec^{-1}(1 : \sqrt{1 - x^2})$.
40. $2 \sec^{-1} x = \tan^{-1}\left[2\sqrt{x^2 - 1} : (2 - x^2)\right]$.
41. $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{1}{4}\pi$.
42. $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{4}{7}$.
43. $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{12}{13} = \sin^{-1} \frac{63}{65}$.
44. $\sin^{-1} \frac{1}{82}\sqrt{82} + \sin^{-1} \frac{4}{41}\sqrt{41} = \frac{1}{4}\pi$.
45. $\sec^{-1} \frac{5}{3} + \sec^{-1} \frac{13}{12} = 75° \, 45'$.
46. $\tan^{-1}(2 + \sqrt{3}) - \tan^{-1}(2 - \sqrt{3}) = \sec^{-1} 2$.
47. $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{1}{4}\pi$
48. $\sin^{-1} x + \sin^{-1}\sqrt{1 - x^2} = \frac{1}{2}\pi$.
49. $\sin^{-1} 0.5 + \sin^{-1} \frac{1}{2}\sqrt{3} = \sin^{-1} 1$.
50. $\tan^{-1} \frac{1}{2} = \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9}$.
51. $\tan^{-1} 0.5 + \tan^{-1} 0.2 + \tan^{-1} 0.125 = \frac{1}{4}\pi$.
52. $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.
53. $\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{10}{11} = \frac{1}{2}\pi$.
54. $\cos^{-1} \frac{3}{10}\sqrt{10} + \sin^{-1} \frac{1}{5}\sqrt{5} = \frac{1}{4}\pi$.

143. Graph of a Function. As in algebra, so in trigonometry, it is possible to represent a function graphically. Before taking up the subject of graphs in trigonometry a few of the simpler cases from algebra will be considered.

Suppose, for example, we have the expression $3x+2$. Since the value of this expression depends upon the value of x, it is called a *function* of x. This fact is indicated by the equation

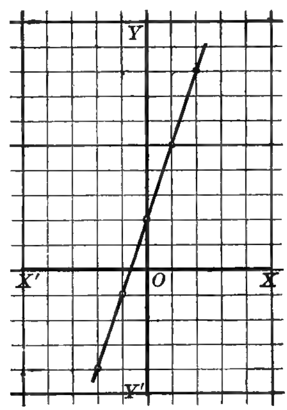
$$f(x)=3x+2,$$

read "function $x = 3x + 2$." But since $f(x)$ is not so easily written as a single letter, it is customary to replace it by some such letter as y, writing the equation

$$y = 3x + 2.$$

If $x=0$, we see that $y=2$; if $x=1$, then $y=5$; and so on. We may form a table of such values, thus:

x	y	x	y
0	2	0	2
1	5	-1	-1
2	8	-2	-4
3	11	-3	-7
⋮	⋮	⋮	⋮



We may then plot the points (0, 2), (1, 5), (2, 8), $\cdots$, $(-1, -1)$, $(-2, -4)$, $\cdots$, as in § 77, and connect them. Then we have the graph of the function $3x+2$.

The graph shows that the function, y or $f(x)$, changes in value much more rapidly than the variable, x. It also shows that the function does not become negative at the same time that the variable does, its value being 2 when $x=0$, and $\frac{1}{2}$ when $x=-\frac{1}{2}$. This kind of function in which x is of the first degree only is called a *linear function* because its graph is a straight line.

Exercise 72. Graphs

Plot the graphs of the following functions:

1. $2x$.
2. $\frac{1}{2}x$.
3. $-x$.
4. $x+1$.
5. $x-1$.
6. $2x+1$.
7. $3-x$.
8. $4-\frac{1}{2}x$.
9. $-2-x$.
10. $2x+3$.
11. $2x-3$.
12. $3-2x$.
13. $0.5x+1.5$.
14. $1.4x-2.3$.
15. $-\frac{15}{4}x-2\frac{1}{2}$.
16. $-\frac{29}{4}x+3\frac{3}{4}$.

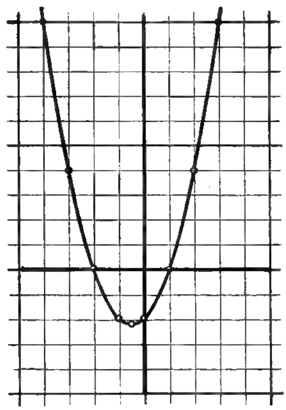
144. Graph of a Quadratic Function. We shall now consider functions of the second degree in the variable. Such a function is called a *quadratic function.*

Taking the function $x^2 + x - 2$, we write

$$y = x^2 + x - 2.$$

Preparing a table of values, as on page 158, we have

x	y	x	y
0	-2	0	-2
1	0	-1	-2
2	4	-2	0
3	10	-3	4
4	18	-4	10
⋮	⋮	⋮	⋮



In order to see where the function lies between $y = -2$ and $y = -2$, we let $x = -\frac{1}{2}$. We find that when $x = -\frac{1}{2}$, $y = -2\frac{1}{4}$. Similarly if we give to x other values between 0 and -1, we shall find that y in every case lies between 0 and -2.

Plotting the points and drawing through them a smooth curve, we have the graph as here shown.

This curve is a *parabola*. All graphs of functions of the form $y = ax^2 + bx + c$ are parabolas.

Graphs of functions of the form $x^2 + y^2 = r^2$, or $y = \pm\sqrt{r^2 - x^2}$, are *circles* with their center at O.

Graphs of functions of the form $a^2x^2 + b^2y^2 = c^2$ are *ellipses*, these becoming circles if $a = b$.

Graphs of functions of the form $a^2x^2 - b^2y^2 = c^2$ are *hyperbolas*.

There are more general equations of all these *conic sections*, but these suffice for our present purposes. The graph of every quadratic function in x and y is always a conic section.

Exercise 73. Graphs of Quadratic Functions

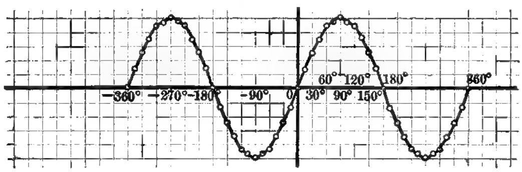
Plot the graphs of the following functions:

1. x^2.	5. $x^2 - 1$.	9. $2x^2 + 3x$.	13. $\pm\sqrt{4 - 3x^2}$.
2. $2x^2$.	6. $x^2 + x + 1$.	10. $3x^2 - 4x$.	14. $\pm\sqrt{5 - 2x^2}$.
3. $\frac{1}{2}x^2$.	7. $x^2 - x + 1$.	11. $\pm\sqrt{4 - x^2}$.	15. $\pm\sqrt{4 + 3x^2}$.
4. $x^2 + 1$.	8. $x^2 + x - 1$.	12. $\pm\sqrt{9 - 4x^2}$.	16. $\pm\sqrt{5 + 2x^2}$.

145. Graph of the Sine. Since $\sin x$ is a function of x, we can plot the graph of $\sin x$. We may represent x, the arc (or angle), in degrees or in radians on the x-axis. Representing it in degrees, as more familiar, we may prepare a table of values as follows:

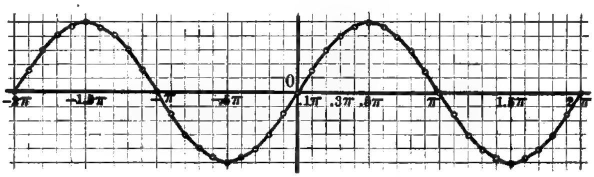
$x =$	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°	⋯
$y =$	0	.26	.5	.7	.87	.97	1	.97	.87	.7	.5	.26	0	⋯

If we represent each unit on the y-axis by $\frac{1}{5}$, and each unit on the x-axis by 30°, the graph is as follows:



The graph shows very clearly that the sine of an angle x is positive between the values $x = 0°$ and $x = 180°$, and also between the values $x = -360°$ and $x = -180°$; that it is negative between the values $x = -180°$ and $x = 0°$, and also between the values $x = 180°$ and $x = 360°$. It also shows that the sine changes from positive to negative as the angle increases and passes through $-180°$ and 180°, and that the sine changes from negative to positive as the angle increases and passes through the values $-360°$, 0°, and 360°. These facts have been found analytically (§ 84), but they are seen more clearly by studying the graph.

If we use radian measure for the arc (angle), and represent each unit on the x-axis by $0.1\,\pi$, the graph is as follows:



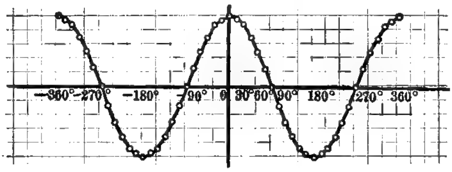
The nature of the curves is the same, the only difference being that we have used different units of measure on the x-axis, thus elongating the curve in the second figure.

146. Periodicity of Functions. This curve shows graphically what we have already found, that periodically the sine comes back to any given value.

Thus $\sin x = 1$ when $x = -270°, 90°, 450°, \cdots$, returning to this value for increase of the angle by every 360°, or $2\,\pi$ radians. The *period* of the sine is therefore said to be 360° or $2\,\pi$.

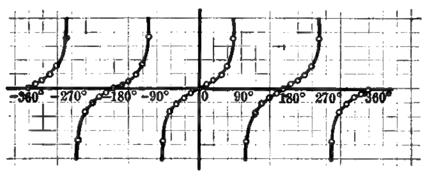
Exercise 74. Graphs of Trigonometric Functions

1. Verify the following plot of the graph of $\cos x$:



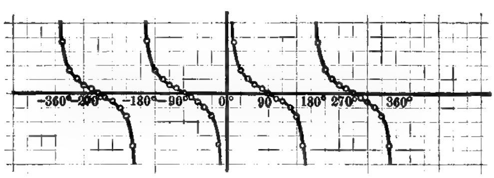
2. What is the period of $\cos x$?

3. Verify the following plot of the graph of $\tan x$:



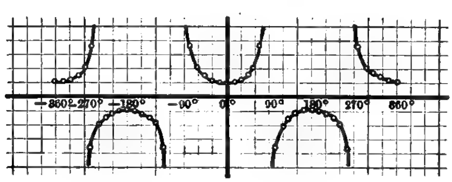
4. What is the period of $\tan x$?

5. Verify the following plot of the graph of $\cot x$:



6. What is the period of $\cot x$?

7. Verify the following plot of the graph of $\sec x$:



8. What is the period of $\sec x$?

9. Plot the graph of $\csc x$, and state the period. Also state at what values of x the sign of $\csc x$ changes.

10. Plot the graphs of $\sin x$ and $\cos x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

Exercise 75. Miscellaneous Exercise

Find the areas of the triangles in which:

1. $a = 25$, $b = 25$, $c = 25$.
2. $a = 25$, $b = 33\frac{1}{3}$, $c = 41\frac{2}{3}$.
3. $a = 74$, $b = 75$, $c = 92$.
4. $a = 2\frac{1}{2}$, $b = 3\frac{1}{3}$, $c = 4\frac{1}{4}$.

5. Consider the area of a triangle with sides 17.2, 26.4, 43.6.

6. Consider the area of a triangle with sides 26.3, 42.4, 73.9.

7. Two inaccessible points A and B are visible from D, but no other point can be found from which both points are visible. Take some point C from which both A and D can be seen and measure CD, 200 ft.; angle ADC, 89°; and angle ACD, 50° 30′. Then take some point E from which both D and B are visible, and measure DE, 200 ft.; angle BDE, 54° 30′; and angle BED, 88° 30′. At D measure angle ADB, 72° 30′. Compute the distance AB.

8. Show by aid of the table of natural sines that $\sin x$ and x agree to four places of decimals for all angles less than 4° 40′.

9. If the values of $\log x$ and $\log \sin x$ agree to five decimal places, find from the table the greatest value x can have.

10. Find four angles whose cosine is the same as the cosine of 175°.

11. Find four angles whose cosine is the same as the cosine of 200°.

12. How many radians in the angle subtended by an arc 7.2 in. long, the radius being 3.6 in.? How many degrees?

13. How many radians in the angle subtended by an arc 1.62 in. long, the radius being 4.86 in.? How many degrees?

Draw the following angles:

14. $-\pi$.
15. -2.
16. $-\frac{1}{2}\pi$.
17. $-\frac{1}{2}$.
18. 2.7π.
19. $2\pi - 6$.
20. $3\pi - 9$.
21. $4 - \pi$.

22. Find four angles whose tangent is $\dfrac{1}{\sqrt{3}}$.

23. Find four angles whose cotangent is $\dfrac{1}{\sqrt{3}}$.

24. Plot the graphs of $\sin x$ and $\csc x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

25. Plot the graphs of $\cos x$ and $\sec x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

26. Plot the graphs of $\tan x$ and $\cot x$ on the same paper. What does the figure tell as to the mutual relation of these functions?

CHAPTER XI

TRIGONOMETRIC IDENTITIES AND EQUATIONS

147. Equation and Identity. An expression of equality which is true for one or more values of the unknown quantity is called an *equation*. An expression of equality which is true for all values of the literal quantities is called an *identity*.

For example, in algebra we may have the equation

$$4x - 3 = 7,$$

which is true only if $x = 2.5$. Or we may have the identity

$$(a + b)^2 = a^2 + 2ab + b^2,$$

which is true whatever values we may give to a and b.

Thus $\sin x = 1$ is a trigonometric equation. It is true for $x = 90°$ or $\frac{1}{2}\pi$, $x = 450°$ or $2\frac{1}{2}\pi$, $x = 810°$ or $4\frac{1}{2}\pi$, and so on, with a period of $360°$ or 2π. In general, therefore, the equation $\sin x = 1$ is true for $x = (2n + \frac{1}{2})\pi$. It is this general value that is required in solving a general trigonometric equation.

On the other hand, the equation $\sin^2 x = 1 - \cos^2 x$ is true for all values of x. It is therefore an identity.

The symbol $\equiv$ is often used instead of $=$ to indicate identity, but the sign of equality is very commonly employed unless special emphasis is to be laid upon the fact that the relation is an identity instead of an ordinary equation.

148. How to prove an Identity. A convenient method of proving a trigonometric identity is to substitute the proper ratios for the functions themselves.

Thus to prove that $\sin x = 1 : \csc x$ we have only to substitute $\frac{a}{c}$ for $\sin x$ and $\frac{c}{a}$ for $\csc x$. We then see that $\frac{a}{c} = 1 : \frac{c}{a}$. Similarly, to prove that $\tan x = \sin x \sec x$, we may substitute $\frac{a}{b}$ for $\tan x$, $\frac{a}{c}$ for $\sin x$, and $\frac{c}{b}$ for $\sec x$. We then have $\frac{a}{b} = \frac{a}{c} \cdot \frac{c}{b}$.

We can often prove a trigonometric identity by reference to formulas already proved.

This was done in proving the identity $\sin 2x = 2 \sin x \cos x$ (§ 101), and in proving $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$ (§ 93).

In some cases it may be better to draw a figure and use a geometric proof, as was done in § 90.

Exercise 76. Identities

Prove the following identities:

1. $\tan x = \dfrac{2\tan\frac{1}{2}x}{1-\tan^2\frac{1}{2}x}.$

2. $\sin x = \dfrac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}.$

3. $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}.$

4. $2\sin x + \sin 2x = \dfrac{2\sin^3 x}{1-\cos x}.$

5. $\sin 3x = \dfrac{\sin^2 2x - \sin^2 x}{\sin x}.$

6. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}.$

7. $\dfrac{\tan 2x + \tan x}{\tan 2x - \tan x} = \dfrac{\sin 3x}{\sin x}.$

8. $\dfrac{3\cos x + \cos 3x}{3\sin x - \sin 3x} = \cot^3 x.$

9. $\dfrac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \cot x.$

10. $\dfrac{\sin 3x + \sin 5x}{\sin x + \sin 3x} = 2\cos 2x.$

11. $\sin x + \sin 3x + \sin 5x = \dfrac{\sin^2 3x}{\sin x}.$

12. $\tan 2x + \sec 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x}.$

13. $\tan x + \tan y = \dfrac{\sin(x+y)}{\cos x \cos y}.$

14. $\tan(x+y) = \dfrac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y}.$

15. $\dfrac{\sin x + \cos y}{\sin x - \cos y} = \dfrac{\tan[\frac{1}{2}(x+y) + 45°]}{\tan[\frac{1}{2}(x-y) - 45°]}.$

16. $\sin 2x + \sin 4x = 2\sin 3x \cos x.$

17. $\sin 4x = 4\sin x\cos x - 8\sin^3 x\cos x.$

18. $\sin 4x = 8\cos^3 x\sin x - 4\cos x\sin x.$

19. $\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x = 1 - 8\sin^2 x + 8\sin^4 x.$

20. $\cos 2x + \cos 4x = 2\cos 3x\cos x.$

21. $\sin 3x - \sin x = 2\cos 2x\sin x.$

22. $\sin^3 x\sin 3x + \cos^3 x\cos 3x = \cos^3 2x.$

23. $\cos^4 x - \sin^4 x = \cos 2x.$

24. $\cos^4 x + \sin^4 x = 1 - \frac{1}{2}\sin^2 2x.$

25. $\cos^6 x - \sin^6 x = (1 - \sin^2 x\cos^2 x)\cos 2x.$

26. $\cos^6 x + \sin^6 x = 1 - 3\sin^2 x\cos^2 x.$

27. $\csc x - 2\cot 2x\cos x = 2\sin x.$

Prove the following identities:

28. $(\sin 2x - \sin 2y)\tan(x+y) = 2(\sin^2 x - \sin^2 y)$.
29. $\sin 3x = 4\sin x \sin(60^\circ + x)\sin(60^\circ - x)$.
30. $\sin 4x = 2\sin x\cos 3x + \sin 2x$.
31. $\sin x + \sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) = 0$.
32. $\cos x \sin(y-z) + \cos y \sin(z-x) + \cos z \sin(x-y) = 0$.
33. $\cos(x+y)\sin y - \cos(x+z)\sin z$
 $= \sin(x+y)\cos y - \sin(x+z)\cos z$.
34. $\cos(x+y+z) + \cos(x+y-z) + \cos(x-y+z)$
 $+ \cos(y+z-x) = 4\cos x\cos y\cos z$.
35. $\sin(x+y)\cos(x-y) + \sin(y+z)\cos(y-z)$
 $+ \sin(z+x)\cos(z-x) = \sin 2x + \sin 2y + \sin 2z$.
36. $\sin(x+y) + \cos(x-y) = 2\sin(x + \frac{1}{4}\pi)\sin(y + \frac{1}{4}\pi)$.
37. $\sin(x+y) - \cos(x-y) = -2\sin(x - \frac{1}{4}\pi)\sin(y - \frac{1}{4}\pi)$.
38. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$.
39. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$.
40. $\sin x + 2\sin 3x + \sin 5x = 4\cos^2 x \sin 3x$.

If A, B, C are the angles of a triangle, prove that:

41. $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.
42. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$.
43. $\sin 3A + \sin 3B + \sin 3C = -4\cos\frac{3}{2}A\cos\frac{3}{2}B\cos\frac{3}{2}C$.
44. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$.

If $A + B + C = 90^\circ$, prove that:

45. $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$.
46. $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A\sin B\sin C$.
47. $\sin 2A + \sin 2B + \sin 2C = 4\cos A\cos B\cos C$.
48. Prove that $\cot^{-1} 3 + \csc^{-1}\sqrt{5} = \frac{1}{4}\pi$.
49. Prove that $x + \tan^{-1}(\cot 2x) = \tan^{-1}(\cot x)$.

Prove the following statements:

50. $\dfrac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \tan 60^\circ$.
51. $\sin 60^\circ + \sin 120^\circ = 2\sin 90^\circ\cos 30^\circ$.
52. $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.
53. $\cos 36^\circ + \sin 36^\circ = \sqrt{2}\cos 9^\circ$.
54. $\tan 11^\circ 15' + 2\tan 22^\circ 30' + 4\tan 45^\circ = \cot 11^\circ 15'$.

149. How to solve a Trigonometric Equation. To solve a trigonometric equation is to find for the unknown quantity the general value which satisfies the equation.

Practically it suffices to find the values between 0° and 360°, since we can then apply our knowledge of the periodicity of the various functions to give us the other values if we need them.

There is no general method applicable to all cases, but the following suggestions will prove of value:

1. *If functions of the sum or difference of two angles are involved, reduce such functions to functions of a single angle.*

Thus, instead of leaving $\sin(x+y)$ in an equation, substitute for $\sin(x+y)$ its equal $\sin x \cos y + \cos x \sin y$.

Similarly, replace $\cos 2x$ by $\cos^2 x - \sin^2 x$, and replace the functions of $\frac{1}{2}x$ by the functions of x.

2. *If several functions are involved, reduce them to the same function.*

This is not always convenient, but it is frequently possible to reduce the equation so as to involve only sines and cosines, or tangents and cotangents, after which the solution can be seen.

3. *If possible, employ the method of factoring in solving the final equation.*

4. *Check the results by substituting in the given equation.*

For example, solve the equation $\cos x = \sin 2x$.

By § 101, $\sin 2x = 2\sin x \cos x$.

$$\therefore \cos x = 2\sin x \cos x.$$

$$\therefore (1 - 2\sin x)\cos x = 0.$$

$$\therefore \cos x = 0, \text{ or } 1 - 2\sin x = 0.$$

$\therefore x = 90°$ or $270°$, $30°$ or $150°$, or these values increased by $2n\pi$.

Each of these values satisfies the given equation.

Exercise 77. Trigonometric Equations

Solve the following equations:

1. $\sin x = 2\sin(\frac{1}{3}\pi + x)$.
2. $\sin 2x = 2\cos x$.
3. $\cos 2x = 2\sin x$.
4. $\sin x + \cos x = 1$.
5. $\sin x + \cos 2x = 4\sin^2 x$.
6. $4\cos 2x + 3\cos x = 1$.
7. $\sin x = \cos 2x$.
8. $\tan x \tan 2x = 2$.
9. $\sec x = 4\csc x$.
10. $\cos\theta + \cos 2\theta = 0$.
11. $\cot\frac{1}{2}\theta + \csc\theta = 2$.
12. $\cot x \tan 2x = 3$.

Solve the following equations:

13. $\sin x + \sin 2x = \sin 3x$.
14. $\sin 2x = 3\sin^2 x - \cos^2 x$.
15. $\cot\theta = \frac{1}{3}\tan\theta$.
16. $2\sin\theta = \cos\theta$.
17. $2\sin^2 x + 5\sin x = 3$.
18. $\tan x \sec x = \sqrt{2}$.
19. $\cos x - \cos 2x = 1$.
20. $\cos 3x + 8\cos^3 x = 0$.
21. $\tan x + \cot x = \tan 2x$.
22. $\tan x + \sec x = a$.
23. $\cos 2x = a(1 - \cos x)$.
24. $\sin^{-1}\frac{1}{2}x = 30°$.
25. $\tan^{-1}x + 2\cot^{-1}x = 135°$.
26. $\sec x - \cot x = \csc x - \tan x$.
27. $\tan 2x \tan x = 1$.
28. $\tan^2 x + \cot^2 x = \frac{10}{3}$.
29. $\sin x + \sin 2x = 1 - \cos 2x$.
30. $4\cos 2x + 6\sin x = 5$.
31. $\sin 4x - \sin 2x = \sin x$.
32. $2\sin^2 x + \sin^2 2x = 2$.
33. $\sin x \sec 2x = 1$.
34. $\sin^2 x + \sin 2x = 1$.
35. $\cos x \sin 2x \csc x = 1$.
36. $\cot x \tan 2x = \sec 2x$.
37. $\sin 2x = \cos 4x$.
38. $\sin 2z \cot z - \sin^2 z = \frac{1}{2}$.
39. $\tan^2 x = \sin 2x$.
40. $\sec 2x + 1 = 2\cos x$.
41. $\tan 2x + \tan 3x = 0$.
42. $\csc x = \cot x + \sqrt{3}$.
43. $\tan x \tan 3x = -\frac{2}{3}$.
44. $\cos 5x + \cos 3x + \cos x = 0$.
45. $\sin^2 x - \cos^2 x = k$.
46. $\sin x + 2\cos x = 1$.
47. $\sin 4x - \cos 3x = \sin 2x$.
48. $\sin x + \cos x = \sec x$.
49. $2\cos x \cos 3x + 1 = 0$.
50. $\cos 3x - 2\cos 2x + \cos x = 0$
51. $\sin(x - 30°) = \frac{1}{2}\sqrt{3}\sin x$.
52. $\sin^{-1}x + 2\cos^{-1}x = \frac{2}{3}\pi$.
53. $\sin^{-1}x + 3\cos^{-1}x = 210°$.
54. $\dfrac{1 - \tan x}{1 + \tan x} = \cos 2x$.
55. $\tan(\frac{1}{4}\pi + x) + \tan(\frac{1}{4}\pi - x) = 4$.
56. $\sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2\cos x$.
57. $\sin(45° + x) + \cos(45° - x) = 1$.
58. $(1 - \tan x)\cos 2x = a(1 + \tan x)$.
59. $\sin^6 x + \cos^6 x = \frac{7}{12}\sin^2 2x$.
60. $\sec(x + 120°) + \sec(x - 120°) = 2\cos x$.
61. $\sin^2 x \cos^2 x - \cos^2 x - \sin^2 x + 1 = 0$.
62. $\sin x + \sin 2x + \sin 3x = 0$.
63. $\sin\theta + 2\sin 2\theta + 3\sin 3\theta = 0$.
64. $\sin 3x = \cos 2x - 1$.
65. $\sin(x + 12°) + \sin(x - 8°) = \sin 20°$.

Solve the following equations:

66. $\tan(60° + x)\tan(60° - x) = -2.$
67. $\tan x + \tan 2x = 0.$
68. $\sin(x + 120°) + \sin(x + 60°) = \frac{3}{2}.$
69. $\sin(x + 30°)\sin(x - 30°) = \frac{1}{2}.$
70. $\sin 2\theta = \cos 3\theta.$
71. $\sin^4 x + \cos^4 x = \frac{5}{8}.$
72. $\sin^4 x - \cos^4 x = \frac{7}{25}.$
73. $\tan(x + 30°) = 2\cos x.$
74. $\sec x = 2\tan x + \frac{1}{4}.$
75. $\sin 11x \sin 4x + \sin 5x \sin 2x = 0.$
76. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0.$
77. $\sin(x + 12°)\cos(x - 12°) = \cos 33° \sin 57$
78. $\sin^{-1} x + \sin^{-1} \frac{1}{2} x = 120°.$
79. $\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3\sqrt{3}.$
80. $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1} 2x.$
81. $(3 - 4\cos^2 x)\sin 2x = 0.$
82. $\cos 2\theta \sec\theta + \sec\theta + 1 = 0.$
83. $\sin x \cos 2x \tan x \cot 2x \sec x \csc 2x = 1.$
84. $\tan(\theta + 45°) = 8\tan\theta.$
85. $\tan(\theta + 45°)\tan\theta = 2.$
86. $\sin x + \sin 3x = \cos x - \cos 3x.$
87. $\sin \frac{1}{2} x(\cos 2x - 2)(1 - \tan^2 x) = 0.$
88. $\tan x + \tan 2x = \tan 3x.$
89. $\cot x - \tan x = \sin x + \cos x.$

Prove the following identities:

90. $(1 + \cot x + \tan x)(\sin x - \cos x) = \dfrac{\sec x}{\csc^2 x} - \dfrac{\csc x}{\sec^2 x}.$
91. $2\csc 2x \cot x = 1 + \cot^2 x.$
92. $\sin a + \sin b + \sin(a + b) = 4\cos\frac{1}{2}a \cos\frac{1}{2}b \sin\frac{1}{2}(a + b).$
93. $\tan(45° + x) - \tan(45° - x) = 2\tan 2x.$
94. $\cot^2 x - \cos^2 x = \cot^2 x \cos^2 x.$
95. $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x.$
96. $\cot^4 x + \cot^2 x = \csc^4 x - \csc^2 x.$
97. $\cos^2 x + \sin^2 x \cos^2 y = \cos^2 y + \sin^2 y \cos^2 x.$

150. Simultaneous Equations. Simultaneous trigonometric equations are solved by the same principles as simultaneous algebraic equations.

1. Required to solve for x and y the system

$$x \sin a + y \sin b = m \qquad (1)$$

$$x \cos a + y \cos b = n \qquad (2)$$

From (1), $x \sin a \cos a + y \sin b \cos a = m \cos a.$ (3)

From (2), $x \sin a \cos a + y \cos b \sin a = n \sin a.$ (4)

From (3) and (4), $y \sin b \cos a - y \cos b \sin a = m \cos a - n \sin a,$

or $$y \sin (b - a) = m \cos a - n \sin a;$$

whence $$y = \frac{m \cos a - n \sin a}{\sin (b - a)}.$$

Similarly, $$x = \frac{n \sin b - m \cos b}{\sin (b - a)}.$$

2. Required to solve for x and y the system

$$\sin x + \sin y = a \qquad (1)$$

$$\cos x + \cos y = b \qquad (2)$$

By § 103, $2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) = a,$ (3)

and $2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y) = b.$

Dividing, $$\tan \tfrac{1}{2}(x + y) = \frac{a}{b}. \qquad (4)$$

$$\therefore \sin \tfrac{1}{2}(x + y) = \frac{a}{\sqrt{a^2 + b^2}}.$$

Substituting the value of $\sin \frac{1}{2}(x + y)$ in (3),

$$\cos \tfrac{1}{2}(x - y) = \tfrac{1}{2}\sqrt{a^2 + b^2}. \qquad (5)$$

From (4), $$x + y = 2 \tan^{-1}\frac{a}{b}. \qquad (6)$$

From (5), $$x - y = 2 \cos^{-1} \tfrac{1}{2}\sqrt{a^2 + b^2}. \qquad (7)$$

From (6) and (7), $$x = \tan^{-1}\frac{a}{b} + \cos^{-1} \tfrac{1}{2}\sqrt{a^2 + b^2},$$

and $$y = \tan^{-1}\frac{a}{b} - \cos^{-1} \tfrac{1}{2}\sqrt{a^2 + b^2}.$$

3. Required to solve for x and y the system

$$y \sin x = a \qquad (1)$$

$$y \cos x = b \qquad (2)$$

Dividing, $$\tan x = \frac{a}{b}.$$

$$\therefore x = \tan^{-1}\frac{a}{b}.$$

Adding the squares of (1) and (2),

$$y^2 (\sin^2 x + \cos^2 x) = a^2 + b^2.$$

Therefore $$y^2 = a^2 + b^2,$$

and $$y = \pm \sqrt{a^2 + b^2},$$

4. Required to solve for x and y the system

$$y \sin(x + a) = m \tag{1}$$

$$y \cos(x + b) = n \tag{2}$$

From (1), $\quad y \sin x \cos a + y \cos x \sin a = m.$

From (2), $\quad y \cos x \cos b - y \sin x \sin b = n.$

We may now solve for $y \sin x$ and $y \cos x$, and then solve for x and y.

5. Required to solve for r, x, and y the system

$$r \cos x \sin y = a \tag{1}$$

$$r \cos x \cos y = b \tag{2}$$

$$r \sin x = c \tag{3}$$

Dividing (1) by (2), $\quad \tan y = \frac{a}{b}.$

$$\therefore y = \tan^{-1}\frac{a}{b}.$$

Squaring (1) and (2) and adding, $\quad r^2 \cos^2 x = a^2 + b^2. \quad (4)$

Taking the square root, $\quad r \cos x = \sqrt{a^2 + b^2}. \quad (5)$

Dividing (3) by (5), $\quad \tan x = \frac{c}{\sqrt{a^2 + b^2}}.$

$$\therefore x = \tan^{-1}\frac{c}{\sqrt{a^2 + b^2}}.$$

Squaring (3) and adding to (4), $\quad r^2 = a^2 + b^2 + c^2.$

$$\therefore r = \sqrt{a^2 + b^2 + c^2}.$$

Exercise 78. Simultaneous Equations

Solve the following systems for x and y:

1. $\sin x + \sin y = \sin a$
 $\cos x + \cos y = 1 + \cos a$

2. $\sin^2 x + \sin^2 y = a$
 $\cos^2 x - \cos^2 y = b$

3. $\sin x - \sin y = 0.7038$
 $\cos x - \cos y = -0.7245$

4. $x \sin 21° + y \cos 44° = 179.70$
 $x \cos 21° + y \sin 44° = 232.30$

5. $\sin^2 x + y = m$
 $\cos^2 x + y = n$

6. $\sin x + \sin y = 1$
 $\sin x - \sin y = 1$

7. $\cos x + \cos y = a$
 $\cos 2x + \cos 2y = b$

8. $\sin x + \sin y = 2m \sin a$
 $\cos x + \cos y = 2n \cos a$

9. Find two angles, x and y, knowing that the sum of their sines is a and the sum of their cosines is b.

Solve the following systems for r and x:

10. $r \sin x = 92.344$
 $r \cos x = 205.309$

11. $r \sin(x - 19° 18') = 59.4034$
 $r \cos(x - 30° 54') = 147.9347$

151. Additional Symbols and Functions. It is the custom in advanced trigonometry and in higher mathematics to represent angles by the Greek letters, and this custom will be followed in the rest of this work where it seems desirable.

The Greek letters most commonly used for this purpose are as follows:

α, alpha	θ, theta
β, beta	λ, lambda
γ, gamma	μ, mu
δ, delta	ϕ, phi
ϵ, epsilon	ω, omega

Besides the six trigonometric functions already studied, there are, as mentioned on page 4, two others that were formerly used and that are still occasionally found in books on trigonometry. These two functions are as follows:

$$\text{versed sine of } \alpha = 1 - \cos\alpha, \text{ written versin } \alpha;$$
$$\text{coversed sine of } \alpha = 1 - \sin\alpha, \text{ written coversin } \alpha.$$

Exercise 79. Simultaneous Equations

1. Solve for ϕ and x:
$$\text{versin}\,\phi = x$$
$$1 - \sin\phi = 0.5$$

2. Solve for θ and x:
$$1 - \sin\theta = x$$
$$1 + \sin\theta = a$$

3. Solve for λ and μ:
$$\sin\lambda = \sqrt{2}\sin\mu$$
$$\tan\lambda = \sqrt{3}\tan\mu$$

4. Solve for θ and ϕ:
$$\sin\theta + \cos\phi = a$$
$$\sin\phi + \cos\theta = b$$

5. Solve for θ and ϕ:
$$a\sin^4\theta - b\sin^4\phi = a$$
$$a\cos^4\theta - b\cos^4\phi = b$$

6. Solve for θ:
$$\sin^2\theta + 2\cos\theta = 2$$
$$\cos\theta - \cos^2\theta = 0$$

152. Eliminant. The equation resulting from the elimination of a certain letter, or of certain letters, between two or more given equations is called the *eliminant* of the given equations with respect to the letter or letters.

For example, if $ax = b$ and $a'x = b'$, it follows by division that $a : a' = b : b'$, or that $ab' = a'b$, and this equality, in which x does not appear, is the eliminant of the given equations with respect to x.

There is no definite rule for discovering the eliminant in trigonometric equations. The study of a few examples and the recalling of identities already considered will assist in the solutions of the problems that arise.

153. Illustrative Examples. The following examples will serve to illustrate the method of finding the eliminant:

1. Find the eliminant, with respect to ϕ, of

$$\sin\phi = a$$
$$\cos\phi = b$$

Since $\sin^2\phi + \cos^2\phi = 1$, we have $a^2 + b^2 = 1$, the eliminant.

2. Find the eliminant, with respect to λ, of

$$\sec\lambda = m$$
$$\tan\lambda = n$$

Since $\sec^2\lambda - \tan^2\lambda = 1$, we have $m^2 - n^2 = 1$, the eliminant.

3. Find the eliminant, with respect to μ, of

$$m\sin\mu + \cos\mu = 1$$
$$n\sin\mu - \cos\mu = 1$$

Writing the equations $m\sin\mu = 1 - \cos\mu$, $n\sin\mu = 1 + \cos\mu$, and multiplying, we have

$$mn\sin^2\mu = 1 - \cos^2\mu = \sin^2\mu.$$

Hence $mn = 1$ is the eliminant.

Exercise 80. Elimination

Find the eliminant with respect to α, θ, λ, μ, or ϕ of the following equations:

1. $\sin\phi + 1 = a$
   $\cos\phi - 1 = b$

2. $\tan\lambda - a = 0$
   $\cot\lambda - b = 0$

3. $\sin\alpha + m = n$
   $\cos\alpha + p = q$

4. $a + \sec\phi = b$
   $p \div \cot\phi = q$

5. $c\sin 2\phi + \cos 2\phi = 1$
   $b\sin 2\phi - \cos 2\phi = 1$

6. $x = r(\theta - \sin\theta)$
   $y = r(1 - \cos\theta)$
   $\theta = \text{versine}^{-1} y/r.$

7. $\sin\phi + \sin 2\phi = m$
   $\cos\phi + \cos 2\phi = n$

8. $a + \sin\theta = \csc\theta$
   $b + \cos\theta = \sec\theta$

9. $\tan\alpha + \sin\alpha = m$
   $\tan\alpha - \sin\alpha = n$

10. $p\sin^2\mu - p\cos^2\mu = r$
    $p'\cos^2\mu - p'\sin^2\mu = r'$

11. $\sin 2\phi + \tan 2\phi = k$
    $\sin 2\phi - \tan 2\phi = l$

12. $p = a\cos\theta\cos\phi$
    $q = b\cos\theta\sin\phi$
    $r = c\sin\theta$

CHAPTER XII

APPLICATIONS OF TRIGONOMETRY TO ALGEBRA

154. Extent of Applications. Trigonometry has numerous applications to algebra, particularly in the approximate solutions of equations and in the interpretation of imaginary roots.

These applications, however, are not essential to the study of spherical trigonometry, and hence this chapter may be omitted without interfering with the student's progress.

For example, if we had no better method of solving quadratic equations we could proceed by trigonometry, and in some cases it is even now advantageous to do so. Consider the equation $x^2 + px - q = 0$. Here the roots are

$$x_1 = -\tfrac{1}{2}p + \tfrac{1}{2}\sqrt{p^2 + 4q}, \quad x_2 = -\tfrac{1}{2}p - \tfrac{1}{2}\sqrt{p^2 + 4q}.$$

If we let $\dfrac{2\sqrt{q}}{p} = \tan\phi$, or $p = 2\sqrt{q}\cot\phi$, we have

$$\begin{aligned} x_1 &= -\sqrt{q}\cot\phi + \sqrt{q}\sqrt{\cot^2\phi + 1} \\ &= -\sqrt{q}\cot\phi + \frac{\sqrt{q}}{\sin\phi} = \sqrt{q}\left(\frac{1}{\sin\phi} - \cot\phi\right) \\ &= \sqrt{q}\,\frac{1 - \cos\phi}{\sin\phi} = \sqrt{q}\tan\tfrac{1}{2}\phi. \end{aligned}$$

Similarly,

$$x_2 = -\sqrt{q}\cot\tfrac{1}{2}\phi.$$

For example, if $x^2 + 1.1102x - 3.3594 = 0$ we have

$$\tan\phi = \frac{2\sqrt{3.3594}}{1.1102};$$

whence $\qquad \log\tan\phi = 0.51876,$

and $\qquad \phi = 73^\circ\, 9'\, 2.6''.$

Therefore $\qquad \log\tan\tfrac{1}{2}\phi = 9.87041 - 10$

and $\qquad \log\sqrt{q} = \log\sqrt{3.3594} = 0.26313.$

Hence $\qquad \log x_1 = 0.13354,$

and $\qquad x_1 \doteq 1.360.$

Similarly, $\qquad x_2 = -2.470.$

155. De Moivre's Theorem. Expressions of the form

$$\cos x + i \sin x,$$

where $i = \sqrt{-1}$, play an important part in modern analysis.

Since $(\cos x + i \sin x)(\cos y + i \sin y)$

$$= \cos x \cos y - \sin x \sin y + i(\cos x \sin y + \sin x \cos y)$$
$$= \cos(x + y) + i \sin(x + y),$$

we have $(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x$;

and again, $(\cos x + i \sin x)^3 = (\cos x + i \sin x)^2 (\cos x + i \sin x)$

$$= (\cos 2x + i \sin 2x)(\cos x + i \sin x)$$
$$= \cos 3x + i \sin 3x.$$

Similarly, $(\cos x + i \sin x)^n = \cos nx + i \sin nx$.

To find the nth power of $\cos x + i \sin x$, n being a positive integer, we have only to multiply the angle x by n in the expression.

This is known as De Moivre's Theorem, from the discoverer (c. 1725).

156. De Moivre's Theorem extended. Again, if n is a positive integer as before,

$$\left(\cos \frac{x}{n} + i \sin \frac{x}{n}\right)^n = \cos x + i \sin x.$$

$$\therefore (\cos x + i \sin x)^{\frac{1}{n}} = \cos \frac{x}{n} + i \sin \frac{x}{n}.$$

However, x may be increased by any integral multiple of 2π without changing the value of $\cos x + i \sin x$. Therefore the following n expressions are the nth roots of $\cos x + i \sin x$:

$$\cos \frac{x}{n} + i \sin \frac{x}{n}, \quad \cos \frac{x + 2\pi}{n} + i \sin \frac{x + 2\pi}{n},$$
$$\cos \frac{x + 4\pi}{n} + i \sin \frac{x + 4\pi}{n}, \cdots,$$
$$\cos \frac{x + (n-1)2\pi}{n} + i \sin \frac{x + (n-1)2\pi}{n}.$$

Hence, if n is a positive integer,

$$(\cos x + i \sin x)^{\frac{1}{n}}$$
$$= \cos \frac{x + 2k\pi}{n} + i \sin \frac{x + 2k\pi}{n} (k = 0, 1, 2, \cdots, n-1).$$

Similarly, it may be shown that

$$(\cos x + i \sin x)^{\frac{m}{n}} = \cos \frac{m}{n}(x + 2k\pi) + i \sin \frac{m}{n}(x + 2k\pi).$$

($k = 0, 1, 2, \cdots, n-1$, m and n being integers, positive or negative.)

157. The Roots of Unity. If we have the binomial equation

$$x^n - 1 = 0,$$

we see that $$x^n = 1,$$

and $$x = \text{the } n\text{th root of } 1,$$

of which the simplest positive root is $\sqrt[n]{1}$ or 1. Since the equation is of the nth degree, there are n roots. In other words, 1 has n nth roots. These are easily found by De Moivre's Theorem.

There are no other roots than those in § 156. For, letting $k = n$, $n + 1$, and so on, we have

$$\cos\frac{x + n(2\pi)}{n} + i\sin\frac{x + n(2\pi)}{n}$$
$$= \cos\left(\frac{x}{n} + 2\pi\right) + i\sin\left(\frac{x}{n} + 2\pi\right) = \cos\frac{x}{n} + i\sin\frac{x}{n},$$

and $$\cos\frac{x + (n+1)2\pi}{n} + i\sin\frac{x + (n+1)2\pi}{n}$$
$$= \cos\left(\frac{x + 2\pi}{n} + 2\pi\right) + i\sin\left(\frac{x + 2\pi}{n} + 2\pi\right)$$
$$= \cos\frac{x + 2\pi}{n} + i\sin\frac{x + 2\pi}{n},$$

and so on, all of which we found when $k = 0, 1, 2, \cdots, n - 1$.

For example, required to find the three cube roots of 1.

If $$\cos\phi + i\sin\phi = 1, \text{ the given number,}$$

then $$\phi = 0, 2\pi, 4\pi, \cdots.$$

Also $$(\cos\phi + i\sin\phi)^{\frac{1}{3}} = 1^{\frac{1}{3}} = \text{the three cube roots of 1.}$$

But $$(\cos\phi + i\sin\phi)^{\frac{1}{3}} = \cos\frac{k(2\pi) + \phi}{3} + i\sin\frac{k(2\pi) + \phi}{3},$$

where $k = 0$, 1, or 2, and $\phi = 0, 2\pi, 4\pi, \cdots$.

Therefore $$1^{\frac{1}{3}} = \cos 2\pi + i\sin 2\pi = 1,$$

or $$1^{\frac{1}{3}} = \cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi = \cos 120^\circ + i\sin 120^\circ$$
$$= -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\cdot i = -0.5 + 0.8660\,i,$$

or $$1^{\frac{1}{3}} = \cos\tfrac{4}{3}\pi + i\sin\tfrac{4}{3}\pi = \cos 240^\circ + i\sin 240^\circ$$
$$= -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\cdot i = -0.5 - 0.8660\,i.$$

The three cube roots of 1 are therefore

$$1, \quad -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \quad -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

These roots could, of course, be obtained algebraically, thus:

$$x^3 - 1 = 0,$$

whence $$(x - 1)(x^2 + x + 1) = 0;$$

and either $$x - 1 = 0, \quad \text{whence } x = 1,$$

or $$x^2 + x + 1 = 0, \quad \text{whence } x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}.$$

Most equations like $x^n - a = 0$ cannot, however, be solved algebraically.

Required to find the seven 7th roots of -1; that is, to solve the equation $x^7 = -1$, or $x^7 + 1 = 0$.

If $\cos\phi + i\sin\phi = -1$, the given number,

then $\phi = \pi, 3\pi, 5\pi, \cdots.$

Also $$(\cos\phi + i\sin\phi)^{\frac{1}{7}} = \cos\frac{k(2\pi)+\phi}{7} + i\sin\frac{k(2\pi)+\phi}{7},$$

where $k = 0, 1, \cdots, 6$, and $\phi = \pi, 3\pi, \cdots$.

That is, in this case
$$(\cos\phi + i\sin\phi)^{\frac{1}{7}} = \cos\frac{(2k+1)\pi}{7} + i\sin\frac{(2k+1)\pi}{7}.$$

Hence the seven 7th roots of 1 are

$$\cos\frac{\pi}{7} + i\sin\frac{\pi}{7} = \cos 25°\,42'\,51\tfrac{3}{7}'' + i\sin 25°\,42'\,51\tfrac{3}{7}'',$$

$$\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7} = \cos 77°\,8'\,34\tfrac{2}{7}'' + i\sin 77°\,8'\,34\tfrac{2}{7}'',$$

and $$\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7},\quad \cos\pi + i\sin\pi,\quad \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7},$$

$$\cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7},\quad \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}.$$

All these values may be found from the tables. For example,
$$\cos 25°\,42'\,51\tfrac{3}{7}'' + i\sin 25°\,42'\,51\tfrac{3}{7}'' = 0.9010 + 0.4339\sqrt{-1}.$$

Exercise 81. Roots of Unity

1. Find by De Moivre's Theorem the two square roots of 1.

2. Find by De Moivre's Theorem the four 4th roots of 1.

3. Find three of the nine 9th roots of 1.

4. Find the five 5th roots of 1.

5. Find the six 6th roots of $+1$ and of -1.

6. Find the four 4th roots of -1.

7. Show that the sum of the three cube roots of 1 is zero.

8. Show that the sum of the five 5th roots of 1 is zero.

9. From Exs. 7 and 8 infer the law as to the sum of the nth roots of 1 and prove this law.

10. From Ex. 9 infer the law as to the sum of the nth roots of k and prove this law.

11. Show that any power of any one of the three cube roots of 1 is one of these three roots.

12. Investigate the law implied in the statement of Ex. 11 for the four 4th roots and the five 5th roots of 1.

158. Roots of Numbers. We have seen that the three cube roots of 1 are

$$x_1 = \cos 120° + i \sin 120° = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3},$$

$$x_2 = \cos 240° + i \sin 240° = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3},$$

and $$x_3 = \cos 360° + i \sin 360° = \cos 0° + i \sin 0° = 1.$$

Furthermore, x_2 is the square of x_1, because

$$(\cos 120° + i \sin 120°)^2 = \cos(2 \cdot 120°) + i \sin(2 \cdot 120°),$$

by De Moivre's Theorem. We may therefore represent the three cube roots by ω, ω^2, and either ω^3 or 1.

In the same way we may represent all n of the nth roots of 1 by $\omega, \omega^2, \omega^3, \cdots, \omega^n$ or 1.

If we have to extract the three cube roots of 8 we can see at once that they are

$$2, \quad 2\,\omega, \quad \text{and} \quad 2\,\omega^2,$$

because $$2^3 = 8, \quad (2\,\omega)^3 = 2^3\,\omega^3 = 8 \cdot 1 = 8,$$

and $$(2\,\omega^2)^3 = 2^3\,\omega^6 = 2^3\,(\omega^3)^2 = 2^3\,1^2 = 8.$$

In general, to find the three cube roots of any number we may take the arithmetical cube root for one of them and multiply this by ω for the second and by ω^2 for the third.

The same is true for any root. For example, if ω, ω^2, ω^3, ω^4, and ω^5 or 1 are the five 5th roots of 1, the five 5th roots of 32 are $2\,\omega$, $2\,\omega^2$, $2\,\omega^3$, $2\,\omega^4$, and $2\,\omega^5$ or 2.

Exercise 82. Roots of Numbers

1. Find the three cube roots of 125.
2. Find the four 4th roots of -81 and verify the results.
3. Find three of the 6th roots of 729 and verify the results.
4. Find three of the 10th roots of 1024 and verify the results.
5. Find three of the 100th roots of 1.
6. Show that, if $2\,\omega$ is one of the complex 7th roots of 128, two of the other roots are $2\,\omega^2$ and $2\,\omega^3$.
7. Show that either of the two complex cube roots of 1 is at the same time the square and the square root of the other.
8. Show that a result similar to the one stated in Ex. 7 can be found with respect to the four 4th roots of 1.
9. Show that the sum of all the nth roots of 1 is zero.
10. Show that the sum of the products of all the nth roots of 1, taken two by two, is zero.

159. Properties of Logarithms. The properties of logarithms have already been studied in Chapter III. These properties hold true whatever base is taken. They are as follows:

1. *The logarithm of 1 is 0.*
2. *The logarithm of the base itself is 1.*
3. *The logarithm of the reciprocal of a positive number is the negative of the logarithm of the number.*
4. *The logarithm of the product of two or more positive numbers is found by adding the logarithms of the several factors.*
5. *The logarithm of the quotient of two positive numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.*
6. *The logarithm of a power of a positive number is found by multiplying the logarithm of the number by the exponent of the power.*
7. *The logarithm of the real positive value of a root of a positive number is found by dividing the logarithm of the number by the index of the root.*

160. Two Important Systems. Although the number of different systems of logarithms is unlimited, there are but two systems which are in common use. These are

1. The common system, also called the Briggs, denary, or decimal system, of which the base is 10.
2. The natural system, of which the base is the fixed value which the sum of the series

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots$$

approaches as the number of terms is indefinitely increased. This base, correct to seven places of decimals, is 2.7182818, and is denoted by the letter e.

Instead of writing $1\cdot 2$, $1\cdot 2\cdot 3$, $1\cdot 2\cdot 3\cdot 4$, and so on, we may write either $2!$, $3!$, $4!$, and so on, or $\underline{|2}$, $\underline{|3}$, $\underline{|4}$, and so on. The expression $2!$ is used on the continent of Europe, $\underline{|2}$ being formerly used in America and England. At present the expression $2!$ is coming to be preferred to $\underline{|2}$ in these two countries.

The common system of logarithms is used in actual calculation; the natural system is used in higher mathematics.

The natural logarithms are also known as Naperian logarithms, in honor of the inventor of logarithms, John Napier (1614), although these are not the ones used by him. They are also known as hyperbolic logarithms.

161. Exponential Series. By the binomial theorem we may expand $\left(1+\frac{1}{n}\right)^{nx}$ and have

$$\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x\left(x-\frac{1}{n}\right)}{2\,!}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3\,!}+\cdots. \qquad (1)$$

This is true for all values of x and n, provided $n>1$. If n is not greater than 1 the series is not *convergent;* that is, the sum approaches no definite limit. The further discussion of convergency belongs to the domain of algebra.

When $x=1$ we have

$$\left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1-\frac{1}{n}}{2\,!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3\,!}+\cdots. \qquad (2)$$

But
$$\left[\left(1+\frac{1}{n}\right)^{n}\right]^{x}=\left(1+\frac{1}{n}\right)^{nx}.$$

Hence, from (1) and (2),

$$\left[1+1+\frac{1-\frac{1}{n}}{2\,!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3\,!}+\cdots\right]^{x}$$
$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{2\,!}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3\,!}+\cdots. \qquad (3)$$

If we take n infinitely large, (3) becomes

$$\left(1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\cdots\right)^{x}=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\cdots; \qquad (4)$$

that is,
$$e^{x}=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\cdots.$$

In particular, if $x=1$ we have

$$e=1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\cdots.$$

We therefore see that we can compute the value of e by simply adding 1, 1, $\frac{1}{2}$ of 1, $\frac{1}{3}$ of $\frac{1}{2}$ of 1, and so on, indefinitely, and that to compute the value to only a few decimal places is a very simple matter. We have merely to proceed as here shown.

	1.000000
2	1.000000
3	0.500000
4	0.166667
5	0.041667
6	0.008333
7	0.001388
8	0.000198
9	0.000025
	0.000003
$e=$	2.71828.

Here we take 1, 1, $\frac{1}{2}$ of 1, $\frac{1}{3}$ of $\frac{1}{2}$ of 1, $\frac{1}{4}$ of $\frac{1}{3}$ of $\frac{1}{2}$ of 1, and so on, and add them. The result given is correct to five decimal places. The result to ten decimal places is 2.7182818284.

162. Expansion of sin *x*, cos *x*, and tan *x*. Denote one radian by 1, and let

$$\cos 1 + i \sin 1 = k.$$

Then $$\cos x + i \sin x = (\cos 1 + i \sin 1)^x = k^x,$$

and, putting $-x$ for x,

$$\cos(-x) + i \sin(-x) = \cos x - i \sin x = k^{-x}.$$

That is, $$\cos x + i \sin x = k^x,$$

and $$\cos x - i \sin x = k^{-x}.$$

By taking the sum and difference of these two equations, and dividing the sum by 2 and the difference by $2i$, we have

$$\cos x = \frac{1}{2}(k^x + k^{-x}),$$

and $$\sin x = \frac{1}{2i}(k^x - k^{-x}).$$

But $k^x = (e^{\log k})^x = e^{x \log k}$, and $k^{-x} = e^{-x \log k}$.

$$\therefore e^{x \log k} = 1 + x \log k + \frac{x^2 (\log k)^2}{2!} + \frac{x^3 (\log k)^3}{3!} + \cdots,$$

and $$e^{-x \log k} = 1 - x \log k + \frac{x^2 (\log k)^2}{2!} - \frac{x^3 (\log k)^3}{3!} + \cdots.$$

$$\therefore \cos x = \frac{1}{2}(k^x + k^{-x}) = 1 + \frac{x^2 (\log k)^2}{2!} + \frac{x^4 (\log k)^4}{4!} + \cdots,$$

and $$\sin x = \frac{1}{i}\left\{x \log k + \frac{x^3 (\log k)^3}{3!} + \frac{x^5 (\log k)^5}{5!} + \cdots\right\}.$$

Dividing the last equation by x, we have

$$\frac{\sin x}{x} = \frac{1}{i}\left\{\log k + \frac{x^2 (\log k)^3}{3!} + \frac{x^4 (\log k)^5}{5!} + \cdots\right\}.$$

But remembering that x represents radians, it is evident that the smaller x is, the nearer $\sin x$ comes to equaling x; that is, the more nearly the sine equals the arc.

Therefore the smaller x becomes, the nearer $\frac{\sin x}{x}$ comes to 1, and the nearer the second member of the equation comes to $\frac{1}{i}\log k$.

We therefore say that, as x approaches the limit 0, the limits of these two members are equal, and

$$1 = \frac{1}{i}\log k;$$

whence $$\log k = i,$$

and $$k = e^i.$$

Therefore, we have

$$\cos x = \frac{1}{2}(e^{xi} + e^{-xi}) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

$$\sin x = \frac{1}{2i}(e^{xi} - e^{-xi}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

From the last two series we obtain, by division,

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}\cdots.$$

By the aid of these series, which rapidly converge, the trigonometric functions of any angle are readily calculated.

In the computation it must be remembered that x is the *circular measure* of the given angle.

Thus to compute $\cos 1$, that is, the cosine of 1 radian or $\cos 57.29578°$, or approximately $\cos 57.3°$, we have

$$\begin{aligned}\cos 1 &= 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots\\ &= 1 - 0.5 + 0.04167 - 0.00139 + 0.00002 - \cdots\\ &= 0.5403 = \cos 57° 18'.\end{aligned}$$

163. Euler's Formula. An important formula discovered in the eighteenth century by the Swiss mathematician Euler will now be considered. We have, as in § 162,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

By multiplying by i in the formula for $\sin x$, we have

$$i \sin x = ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots.$$

Adding,

$$\cos x + i \sin x = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots.$$

By substituting ix for x in the formula for e^x, we see that

$$\begin{aligned}e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots\\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots.\end{aligned}$$

In other words,

$$e^{ix} = \cos x + i \sin x.$$

164. Deductions from Euler's Formula. Euler's formula is one of the most important formulas in all mathematics. From it several important deductions will now be made.

Since $e^{ix} = \cos x + i \sin x$, in which x may have any values, we may let $x = \pi$. We then have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0,$$

or

$$e^{i\pi} = -1.$$

In this formula we have combined four of the most interesting numbers of mathematics, e (the natural base), i (the imaginary unit, $\sqrt{-1}$), π (the ratio of the circumference to the diameter), and -1 (the negative unit).

Furthermore, we see that a real number (e) may be affected by an imaginary exponent ($i\pi$) and yet have the power real (-1).

Taking the square root of each side of the equation $e^{i\pi} = -1$, we have

$$e^{\frac{i\pi}{2}} = \sqrt{-1} = i.$$

Taking the logarithm of each side of the equation $e^{i\pi} = -1$, we have

$$i\pi = \log(-1).$$

Hence we see that -1 has a logarithm, but that it is an imaginary number and is, therefore, not suitable for purposes of calculation.

Since $\cos \phi + i \sin \phi = \cos(2k\pi + \phi) + i \sin(2k\pi + \phi)$, we see that $e^{\phi i}$, which is equal to $\cos \phi + i \sin \phi$, may be written $e^{(2k\pi + \phi)i}$, or we may write

$$e^{\phi i} = e^{(2k\pi + \phi)i} = \cos \phi + i \sin \phi = \cos(2k\pi + \phi) + i \sin(2k\pi + \phi).$$

Hence $\quad (2k\pi + \phi)i = \log[\cos(2k\pi + \phi) + i \sin(2k\pi + \phi)].$

If $\phi = 0$, $\quad 2k\pi i = \log 1.$

If $k = 0$, this reduces to $0 = \log 1$.

If $k = 1$ we have $2\pi i = \log 1$.; if $k = 2$, we have $4\pi i = \log 1$, and so on. In other words, $\log 1$ is multiple-valued, but only one of these values is real.

If $\phi = \pi$, $\quad (2k\pi + \pi)i = (2k + 1)\pi i = \log(-1).$

Hence the logarithms of negative numbers are always imaginary; for if $k = 0$ we have $\pi i = \log(-1)$; if $k = 1$ we have $3\pi i = \log(-1)$; and so on.

If we wish to consider the logarithm of some number N, we have

$$Ne^{2k\pi i} = N(\cos 2k\pi + i \sin 2k\pi).$$

Hence

$$\begin{aligned} \log N + 2k\pi i &= \log N + \log(\cos 2k\pi + i \sin 2k\pi) \\ &= \log N + \log 1 = \log N. \end{aligned}$$

That is, $\log N = \log N + 2k\pi i$. Hence the logarithm of a number is the logarithm given by the tables, $+ 2k\pi i$. If $k = 0$ we have the usual logarithm, but for other values of k we have imaginaries.

Exercise 83. Properties of Logarithms

Prove the following properties of logarithms as given in § 159, using b as the base:

1. Properties 1 and 2.
2. Property 3.
3. Property 4.
4. Property 5.
5. Property 6.
6. Property 7.

Find the value of each of the following:

7. 5!
8. 7!
9. 6!
10. 8!
11. 10!

Simplify the following:

12. $\frac{10!}{3!}$.
13. $\frac{10!}{8!}$.
14. $\frac{7!}{5!}$.
15. $\frac{15!}{14!}$.
16. $\frac{20!}{17!}$.

17. Find to five decimal places the value of $\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)^2$.

18. Find to five decimal places the value of $\left(2+\frac{1}{2!}+\frac{1}{3!}+\cdots\right)^{\frac{1}{2}}$.

By the use of the series for $\cos x$ *find the following:*

19. $\cos \frac{1}{2}$.
20. $\cos \frac{1}{8}$.
21. $\cos 2$.
22. $\cos 0$.

By the use of the series for $\sin x$ *find the following:*

23. $\sin 1$.
24. $\sin \frac{1}{2}$.
25. $\sin 2$.
26. $\sin 0$.

By the use of the series for $\tan x$ *find the following:*

27. $\tan 0$.
28. $\tan 1$.
29. $\tan \frac{1}{2}$.
30. $\tan 2$.

Prove the following statements:

31. $e^{2\pi i} = 1$.
32. $e^{-\frac{\pi}{2}} = i^i$.
33. $e^\pi = \sqrt[i]{-1}$.
34. $e^i = \sqrt[\pi]{-1}$.

Given $\log_e 2 = 0.6931$, *find two logarithms (to the base e) of:*

35. 2.
36. 4.
37. $\sqrt{2}$.
38. -2.

Given $\log_e 5 = 1.609$, *find three logarithms (to the base e) of:*

39. 5.
40. 25.
41. 125.
42. -5.

Given $\log_e 10 = 2.302585$, *find two logarithms (to the base e) of:*

43. 100.
44. -10.
45. 1000.
46. $\sqrt{10}$.

47. From the series of § 162 show that $\sin(-\phi) = -\sin\phi$.

48. Prove that the ratio of the circumference of a circle to the diameter equals $-2\log(i^i) = -2i\log i$.

Exercise 84. Review Problems

1. The angle of elevation of the top of a vertical cliff at a point 575 ft. from the foot is $32° 15'$. Find the height of the cliff.

2. An aeroplane is above a straight road on which are two observers 1640 ft. apart. At a given signal the observers take the angles of elevation of the aeroplane, finding them to be $58°$ and $63°$ respectively. Find the height of the aeroplane and its distance from each observer.

3. Prove that $\left(\sqrt{\csc x + \cot x} - \sqrt{\csc x - \cot x}\right)^2 = 2(\csc x - 1)$.

4. Given $\sin x = 2m/(m^2+1)$ and $\sin y = 2n/(n^2+1)$, find the value of $\tan(x+y)$.

5. Find the least value of $\cos^2 x + \sec^2 x$.

6. Prove that $1 - \sin^2 x/\sin^2 y = \cos^2 x(1 - \tan^2 x/\tan^2 y)$.

7. Prove this formula, due to Euler: $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{1}{4}\pi$.

8. Prove that $\cot\frac{1}{2}x - \cot x = \csc x$.

9. Prove that $(\sin x + i\cos x)^n = \cos n(\frac{1}{2}\pi - x) + i\sin n(\frac{1}{2}\pi - x)$.

10. Show that $\log i = \frac{1}{2}\pi i$ and that $\log(-i) = -\frac{1}{2}\pi i$.

11. Through the excenters of a triangle ABC lines are drawn parallel to the three sides, thus forming another triangle $A'B'C'$. Prove that the perimeter of $\triangle A'B'C'$ is $4r\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C$, where r is the radius of the circumcircle.

12. Given two sides and the included angle of a triangle, find the perpendicular drawn to the third side from the opposite vertex.

13. To find the height of a mountain a north-and-south base line is taken 1000 yd. long. From one end of this line the summit bears N. 80° E., and has an angle of elevation of $13° 14'$; from the other end it bears N. 43° 30′ E. Find the height of the mountain.

14. The angle of elevation of a wireless telegraph tower is observed from a point on the horizontal plain on which it stands. At a point a feet nearer, the angle of elevation is the complement of the former. At a point b feet nearer still, the angle of elevation is double the first. Show that the height of the tower is $[(a+b)^2 - \frac{1}{4}a^2]^{\frac{1}{2}}$.

Prove the following formulas:

15. $2\cos^2 x = \cos 2x + 1$.

16. $2\sin^2 x = -\cos 2x + 1$.

17. $8\cos^4 x = \cos 4x + 4\cos 2x + 3$

18. $4\cos^3 x = \cos 3x + 3\cos x$.

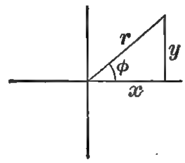
19. $4\sin^3 x = -\sin 3x + 3\sin x$.

20. $8\sin^4 x = \cos 4x - 4\cos 2x + 3$.

THE MOST IMPORTANT FORMULAS OF PLANE TRIGONOMETRY

Right Triangles (§§ 15–21)

1. $y = r \sin\phi$.
2. $x = r\cos\phi$.
3. $y = x\tan\phi$.
4. $x = y\cot\phi$.
5. $r = x\sec\phi$.
6. $r = y\csc\phi$.



Relations of Functions (§§ 13, 14, 89)

7. $\sin\phi = \dfrac{1}{\csc\phi}$.
8. $\cos\phi = \dfrac{1}{\sec\phi}$.
9. $\tan\phi = \dfrac{1}{\cot\phi}$.
10. $\sin\phi\csc\phi = 1$.
11. $\cos\phi\sec\phi = 1$.
12. $\cot\phi = \dfrac{1}{\tan\phi}$.
13. $\sec\phi = \dfrac{1}{\cos\phi}$.
14. $\csc\phi = \dfrac{1}{\sin\phi}$.
15. $\tan\phi\cot\phi = 1$.
16. $\sin^2\phi + \cos^2\phi = 1$.
17. $\sin\phi = \dfrac{\cos\phi}{\cot\phi}$.
18. $\tan\phi = \dfrac{\sin\phi}{\cos\phi}$.
19. $\cot\phi = \dfrac{\cos\phi}{\sin\phi}$.
20. $1 + \tan^2\phi = \sec^2\phi$.
21. $1 + \cot^2\phi = \csc^2\phi$.

Functions of $x \pm y$ (§§ 90–100)

22. $\sin(x+y) = \sin x\cos y + \cos x\sin y$.
23. $\sin(x-y) = \sin x\cos y - \cos x\sin y$.
24. $\cos(x+y) = \cos x\cos y - \sin x\sin y$.
25. $\cos(x-y) = \cos x\cos y + \sin x\sin y$.
26. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$.
27. $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x\tan y}$.
28. $\cot(x+y) = \dfrac{\cot x\cot y - 1}{\cot y + \cot x}$.
29. $\cot(x-y) = \dfrac{\cot x\cot y + 1}{\cot y - \cot x}$.

Functions of Twice an Angle (§ 101)

30. $\sin 2\phi = 2\sin\phi\cos\phi$.
31. $\tan 2\phi = \dfrac{2\tan\phi}{1-\tan^2\phi}$.
32. $\cos 2\phi = \cos^2\phi - \sin^2\phi$.
33. $\cot 2\phi = \dfrac{\cot^2\phi - 1}{2\cot\phi}$.

Functions of Half an Angle (§ 102)

34. $\sin\frac{1}{2}\phi = \pm\sqrt{\dfrac{1-\cos\phi}{2}}$.
35. $\cos\frac{1}{2}\phi = \pm\sqrt{\dfrac{1+\cos\phi}{2}}$.
36. $\tan\frac{1}{2}\phi = \pm\sqrt{\dfrac{1-\cos\phi}{1+\cos\phi}}$.
37. $\cot\frac{1}{2}\phi = \pm\sqrt{\dfrac{1+\cos\phi}{1-\cos\phi}}$.

Functions involving Half Angles (§ 101)

38. $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}.$

39. $\tan x = \dfrac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}.$

40. $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}.$

41. $\cot x = \dfrac{\cot^2 \frac{x}{2} - 1}{2 \cot \frac{x}{2}}.$

Sums and Differences of Functions (§ 103)

42. $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$

43. $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$

44. $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$

45. $\cos A - \cos B = - 2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$

46. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$

Laws of Sines, Cosines, and Tangents (§§ 105, 111, 112)

47. Law of sines, $\dfrac{a}{b} = \dfrac{\sin A}{\sin B},$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

48. Law of cosines, $a^2 = b^2 + c^2 - 2\,bc \cos A.$

49. Law of tangents, $\dfrac{a - b}{a + b} = \dfrac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}$, if $a > b$;

$$\frac{b - a}{b + a} = \frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)}, \text{ if } a < b.$$

Formulas in Terms of Sides (§§ 115, 116)

50. $\dfrac{a + b + c}{2} = s.$

51. $\sin \frac{1}{2} A = \sqrt{\dfrac{(s - b)(s - c)}{bc}}.$

52. $\cos \frac{1}{2} A = \sqrt{\dfrac{s(s - a)}{bc}}.$

53. $\sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}} = r.$

54. $\tan \frac{1}{2} A = \sqrt{\dfrac{(s - b)(s - c)}{s(s - a)}}.$

55. $\tan \frac{1}{2} A = \dfrac{r}{s - a}.$

Areas of Triangles (§ 118)

56. Area of triangle $ABC = \frac{1}{2}\, ac \sin B = \frac{1}{2} r(a + b + c) = rs = \sqrt{s(s - a)(s - b)(s - c)} = \dfrac{abc}{4R} = \dfrac{a^2 \sin B \sin C}{2 \sin (B + C)}.$

SPHERICAL TRIGONOMETRY

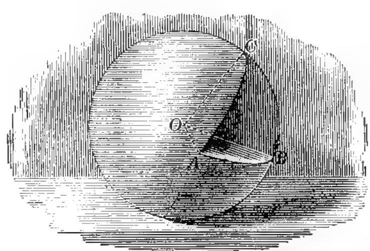
CHAPTER I

THE RIGHT SPHERICAL TRIANGLE

165. Spherical Triangle. A portion of a spherical surface bounded by three arcs of great circles is called a *spherical triangle.*

The bounding arcs are called the *sides* of the triangle, the angles between the sides are called the *angles* of the triangle, and the points of intersection of the sides are called the *vertices* of the triangle.

166. Relation of Spherical Triangles to Trihedral Angles. The planes of the sides of a spherical triangle form a trihedral angle whose vertex is the center of the sphere, whose face angles are measured by the sides of the triangle, and whose dihedral angles have the same numerical measure as the angles of the triangle.



Thus the planes of the sides of the spherical triangle ABC form the trihedral angle $O\text{-}ABC$. The face angles AOB, BOC, and COA of the trihedral angle are measured by the sides AB, BC, and CA of the spherical triangle. The dihedral angle whose edge is OA has the same measure as the spherical angle BAC, and so on.

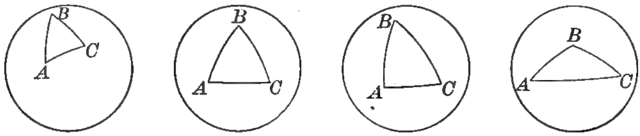
Hence from any property of trihedral angles we may infer an analogous property of spherical triangles; and conversely.

The sides of the triangle may have any value from 0° to 360°; but in this work only sides that are less than 180° will be considered. The angles may have any value from 0° to 180°.

167. Spherical Trigonometry. The solution of spherical triangles is the chief object of *spherical trigonometry.*

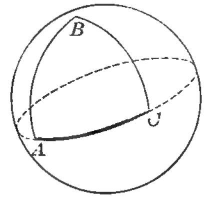
In Plane Trigonometry it was shown that any plane triangle can be solved if three independent parts are given. In Spherical Trigonometry it will be shown that any spherical triangle on a given sphere can be solved if any three of its six parts are given, even though these given parts are the three angles,

168. Spherical Triangles Classified. A spherical triangle may be *right*, *obtuse*, or *acute*. It may also be *equilateral*, *equiangular*, *scalene*, or *isosceles*. These terms are used as in the case of plane triangles.



Isosceles Equilateral Right Obtuse

When a spherical triangle has one or more of its sides equal to a quadrant, it is called a *quadrantal triangle*.



A spherical triangle, unlike a plane triangle, may have two or even three right angles, as is seen in the case of the quadrantal triangle here shown.

Furthermore, it is evident that angle B may increase to the limit 180°, and that angles A and C may also increase in the same way, the limit of the sum of angles A, B, and C being 540°.

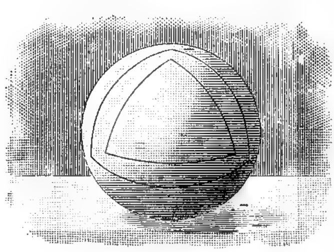
169. Geometric Properties of Spherical Triangles. The following properties of spherical triangles are proved in geometry:

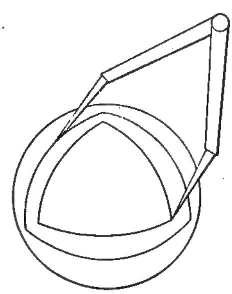
1. *Any side of a spherical triangle is less than the sum of the other two sides.*

2. *If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the greater side is opposite the greater angle; and conversely.*

3. *The sum of the sides of a spherical triangle is less than 360°.*

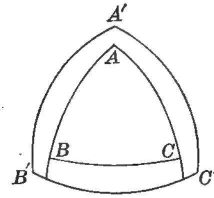
4. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*





5. *If, from the vertices of a spherical triangle as poles, arcs of great circles are described, another triangle is formed so related to the first that each angle of either triangle is the supplement of the side opposite it in the other triangle.*

170. Polar Triangle. As stated in § 169, if arcs of great circles are described from the vertices of a spherical triangle as poles, another spherical triangle is formed which is called the *polar triangle* of the first.



Thus, if A is the pole of the arc $B'C'$ of a great circle, B the pole of arc $C'A'$, C the pole of arc $A'B'$, then $A'B'C'$ is the polar triangle of ABC.

If, with A, B, C as poles, entire great circles are described, these circles divide the surface of the sphere into *eight* spherical triangles as can easily be seen by describing the circles on a wooden ball.

Of these eight triangles, that one is the polar of ABC whose vertex A', corresponding to A, lies on the same side of BC as the vertex A; and similarly for the other vertices.

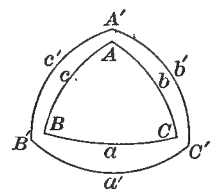
It is desirable in the study of spherical trigonometry, and particularly in the study of polar triangles, to have a spherical blackboard. When this is not available, any wooden ball will serve the purpose. With such aids the polar triangle is much more clearly understood.

171. Properties of Polar Triangles. It is shown in geometry, as stated in § 169 and § 170, that:

1. *If one spherical triangle is the polar triangle of another, then reciprocally the second is the polar triangle of the first.*

2. *In two polar triangles each angle of one is the supplement of the opposite side in the other.*

That is, in this figure,



$$A + a' = 180^\circ, \qquad A' + a = 180^\circ,$$
$$B + b' = 180^\circ, \qquad B' + b = 180^\circ,$$
$$C + c' = 180^\circ, \qquad C' + c = 180^\circ.$$

These statements may be written

$$A = 180^\circ - a', \qquad a' = 180^\circ - A, \qquad A' = 180^\circ - a,$$
$$B = 180^\circ - b', \qquad b' = 180^\circ - B, \qquad B' = 180^\circ - b,$$
$$C = 180^\circ - c', \qquad c' = 180^\circ - C, \qquad \text{and so on.}$$

Therefore, if the angles of a spherical triangle are 59° 20′, 86° 40′, and 78° 50′ respectively, the opposite sides of the polar triangle are 120° 40′, 93° 20′, and 101° 10′ respectively; and if the sides of a spherical triangle are 82° 10′, 112° 20′, and 74° 40′ respectively, the opposite angles of the polar triangle are 107° 50′, 67° 40′, and 105° 20′ respectively. Thus we see that if we can solve a spherical triangle, we can solve its polar triangle, and vice versa, a fact of which we shall make great use in the subsequent work in spherical trigonometry.

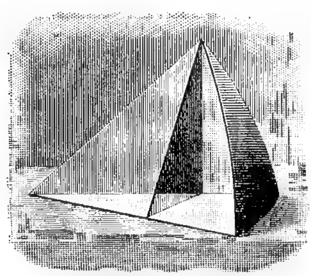
172. Formulas of the Right Triangle. It can easily be shown by elementary geometry that the following theorems are true:

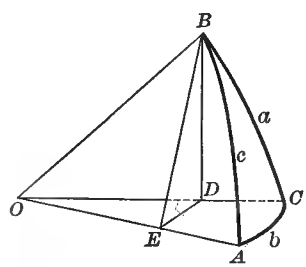
1. *If a spherical triangle has three right angles, the sides of the triangle are quadrants.*

2. *If a spherical triangle has two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the opposite side.*

When we say that an angle is measured by an arc the same meaning is to be assigned as in geometry; that is, the number of degrees in the angle is equal to the number of degrees in the arc.

Therefore, if a right triangle has three right angles, we have the solution at once, from the first of these theorems, for each side is then a quadrant; and if a triangle has two right angles and the included side given, we have the solution from the second theorem, for two sides are quadrants and the third angle is measured by the given opposite side. Hence we need to consider right triangles having only one right angle.





Let $\triangle ACB$ be a right spherical triangle, with C the right angle, and with A and B not right angles.

We shall, for the present, suppose all the parts except C to be less than 90°, and the radius of the sphere to be 1. Other cases will be considered in § 173, and it will be found that the formulas here deduced are general.

Construct the corresponding trihedral angle $O\text{–}ACB$.

Pass a plane through B perpendicular to OA, and let it intersect the faces of the trihedral angle $O\text{–}ACB$ in ED, DB, and BE.

It will be seen from the above figure that the parts of the dihedral angle are now separated into simpler elements, which we can study in the light of plane trigonometry.

This plan might also be taken in the study of other spherical triangles, but it is more convenient to break them up into right triangles and refer back to this section.

In these figures we see that

BE is perpendicular to OA, and DE is perpendicular to OA.

(*For OA is perpendicular to the plane EDB.*)

$$\therefore \angle DEB = \angle A.$$

(*For each has the same measure as the dihedral angle.*)

Also, plane EDB is perpendicular to plane AOC,

(*If a line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*)

and plane COB is perpendicular to plane AOC,

(*Because $\angle C$ is given as a right angle.*)

$\therefore$ BD is perpendicular to plane AOC.

(*If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*)

$\therefore$ BD is perpendicular to OC and DE.

Since $\cos c = OE = OD \cos b$, and $OD = \cos a$,

$$\therefore \mathbf{\cos c = \cos a \cos b.} \tag{1}$$

Since $\sin a = BD = BE \sin A$, and $BE = \sin c$,

$$\therefore \mathbf{\sin a = \sin c \sin A.} \tag{2}$$

Similarly, $$\mathbf{\sin b = \sin c \sin B.} \tag{3}$$

This may be found by passing a plane through A perpendicular to OB, but it is apparent by merely interchanging A and B, a and b.

Since $$\cos A = \frac{DE}{BE} = \frac{OE \tan b}{OE \tan c},$$

$$\therefore \mathbf{\cos A = \tan b \cot c.} \tag{4}$$

Similarly, $$\mathbf{\cos B = \tan a \cot c.} \tag{5}$$

Since $$\cos A = \frac{DE}{BE} = \frac{OD \sin b}{\sin c} = \cos a \frac{\sin b}{\sin c} = \cos a \frac{\sin c \sin B}{\sin c},$$

$$\therefore \mathbf{\cos A = \cos a \sin B.} \tag{6}$$

Similarly, $$\mathbf{\cos B = \cos b \sin A.} \tag{7}$$

Since $$\sin b = \frac{DE}{OD} = \frac{BD \cot A}{OD} = \tan a \cot A,$$

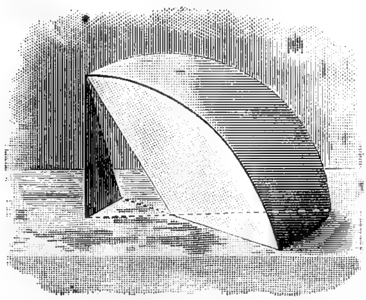
$$\therefore \mathbf{\sin b = \tan a \cot A.} \tag{8}$$

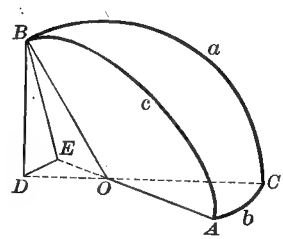
Similarly, $$\mathbf{\sin a = \tan b \cot B.} \tag{9}$$

Substituting in (1) the values of $\cos a$ and $\cos b$ found from (6) and (7), we have $$\mathbf{\cos c = \cot A \cot B.} \tag{10}$$

173. The Proofs Extended. The ten formulas of § 172 are sufficient for the solution of any right spherical triangle. For simplicity in deducing these formulas all the parts of the triangle, except the right angle, were assumed to be less than 90°. But the formulas are entirely general and hold for all types of right triangle, whatever may be the size of the parts.

For example, suppose that one of the sides a, of the right triangle, is greater than 90°, and construct a figure for this case in the same manner as on page 190.





The auxiliary plane BDE will now cut both CO and AO produced beyond the center O; and we have

$$\begin{aligned}\cos c &= -OE = -OD\cos DOE\\ &= -(-\cos a)\cos b\\ &= \cos a\cos b.\end{aligned}$$

Similarly, $$\begin{aligned}\sin a &= BD = BE\sin A\\ &= \sin c\sin A,\end{aligned}$$

exactly as in § 172.

Likewise, the other eight formulas on page 191 hold true in case either side is greater than 90°.

Again, suppose that both the sides a and b are greater than 90°. In this case the plane BDE will cut CO produced beyond O, and AO between A and O; and we have

$$\begin{aligned}\cos c &= OE = OD\cos DOE\\ &= (-\cos a)(-\cos b)\\ &= \cos a\cos b,\end{aligned}$$

exactly as in § 172.

Likewise the other formulas on page 191 hold true in this case.

Like results may be obtained in all cases.

In other words, the ten formulas in § 172 are universally true.

174. The Formulas Extended. From the ten formulas given on page 191 numerous others can be deduced. The ten formulas will now be restated and certain of the most important deductions will be made.

1. $\cos c = \cos a \cos b$.

Dividing by $\cos a$ and reducing, we have $\cos b = \cos c \sec a$. Similarly, we may divide by $\cos b$ and then have $\cos a = \cos c \sec b$, but of course we can get this formula by merely interchanging a and b.

2. $\sin a = \sin c \sin A$.

Dividing by $\sin c$ and reducing, we have $\sin A = \sin a \csc c$.

Dividing by $\sin A$ and reducing, we have $\sin c = \sin a \csc A$. Of course in all formulas containing $\sec x$ or $\csc x$ we may use $\dfrac{1}{\cos x}$ and $\dfrac{1}{\sin x}$ in place of $\sec x$ and $\csc x$. Indeed, as we have found, in computation with logarithms it is as easy to use the latter forms, and the secant and cosecant are of little practical value because of this fact.

3. $\sin b = \sin c \sin B$.

4. $\cos A = \tan b \cot c$.

Dividing by $\cot c$ and reducing, we have $\tan b = \tan c \cos A$. Similarly, we may divide by $\tan b$ and then have $\cot c = \cot b \cos A$.

5. $\cos B = \tan a \cot c$.

Dividing by $\cos B$ and $\cot c$ and reducing, we have $\tan c = \tan a \sec B$. It is evident that we can derive various other formulas from this one.

6. $\cos A = \cos a \sin B$.

Dividing by $\cos a$ and reducing, we have $\sin B = \sec a \cos A$.

Dividing by $\sin B$ and reducing, we have $\cos a = \cos A \csc B$.

7. $\cos B = \cos b \sin A$.

Dividing by $\sin A$ and reducing, we have $\cos b = \cos B \csc A$. Similarly, we can obtain other formulas by dividing by $\cos b$ or by $\cos B$.

8. $\sin b = \tan a \cot A$.

Dividing by $\sin b$ and $\cot A$ and reducing, we have $\tan A = \tan a \csc b$.

Interchanging A and B, and a and b, we have $\tan B = \tan b \csc a$.

9. $\sin a = \tan b \cot B$.

Dividing by $\cot B$ and reducing, we have $\tan b = \sin a \tan B$.

10. $\cos c = \cot A \cot B$.

Dividing by $\cot A$ and reducing, we have $\cot B = \cos c \tan A$.

Sometimes it is easier to use these deduced formulas than to use the ten from which they are derived. For example, suppose c and A are given, to find B. We might substitute in Formula 10 on page 191 and solve for $\cot B$, but if we use the formula $\cot B = \cos c \tan A$, the solution is already effected. It is not necessary to remember the derived formulas, however.

175. Auxiliary Formulas. The following auxiliary formulas may be used occasionally when small angles are involved.

1. $\tan^2 \frac{1}{2} b = \tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a)$.

We have $\tan^2 \frac{1}{2} b = \dfrac{1 - \cos b}{1 + \cos b} = \dfrac{1 - \dfrac{\cos b}{\cos a}}{1 + \dfrac{\cos c}{\cos a}} = \dfrac{\cos a - \cos c}{\cos a + \cos c}$

$$= \frac{-2 \sin \frac{1}{2}(a + c) \sin \frac{1}{2}(a - c)}{2 \cos \frac{1}{2}(a + c) \cos \frac{1}{2}(a - c)}$$
$$= -\tan \tfrac{1}{2}(a + c) \tan \tfrac{1}{2}(a - c) = \tan \tfrac{1}{2}(c + a) \tan \tfrac{1}{2}(c - a).$$

2. $\tan^2(45° - \frac{1}{2} A) = \tan \frac{1}{2}(c - a) \cot \frac{1}{2}(c + a)$.

We have $\tan^2(45° - \frac{1}{2} A) = \tan^2 \frac{1}{2}(90° - A) = \cot^2 \frac{1}{2}(90° + A) = \dfrac{1 + \cos(90° + A)}{1 - \cos(90° + A)}$

$$= \frac{1 - \sin A}{1 + \sin A} = \frac{1 - \dfrac{\sin a}{\sin c}}{1 + \dfrac{\sin a}{\sin c}} = \frac{\sin c - \sin a}{\sin c + \sin a} = \frac{2 \cos \frac{1}{2}(c + a) \sin \frac{1}{2}(c - a)}{2 \sin \frac{1}{2}(c + a) \cos \frac{1}{2}(c - a)}$$
$$= \tan \tfrac{1}{2}(c - a) \cot \tfrac{1}{2}(c + a).$$

3. $\tan^2 \frac{1}{2} B = \dfrac{\sin(c - a)}{\sin(c + a)}$.

We have $\tan^2 \frac{1}{2} B = \dfrac{1 - \cos B}{1 + \cos B} = \dfrac{1 - \dfrac{\tan a}{\tan c}}{1 + \dfrac{\tan a}{\tan c}} = \dfrac{\tan c - \tan a}{\tan c + \tan a} = \dfrac{\dfrac{\sin c}{\cos c} - \dfrac{\sin a}{\cos a}}{\dfrac{\sin c}{\cos c} + \dfrac{\sin a}{\cos a}}$

$$= \frac{\sin c \cos a - \cos c \sin a}{\sin c \cos a + \cos c \sin a} = \frac{\sin(c - a)}{\sin(c + a)}.$$

4. $\tan^2 \frac{1}{2} c = \dfrac{-\cos(A + B)}{\cos(A - B)}$.

We have $\tan^2 \frac{1}{2} c = \dfrac{1 - \cos c}{1 + \cos c} = \dfrac{1 - \dfrac{\cot A}{\tan B}}{1 + \dfrac{\cot A}{\tan B}} = \dfrac{\tan B - \cot A}{\tan B + \cot A} = \dfrac{\dfrac{\sin B}{\cos B} - \dfrac{\cos A}{\sin A}}{\dfrac{\sin B}{\cos B} + \dfrac{\cos A}{\sin A}}$

$$= \frac{\sin A \sin B - \cos A \cos B}{\sin A \sin B + \cos A \cos B} = \frac{-\cos(A + B)}{\cos(A - B)}.$$

5. $\tan^2 \frac{1}{2} a = \tan[\frac{1}{2}(A + B) - 45°] \tan[\frac{1}{2}(A - B) + 45°]$.

We have $\tan^2 \frac{1}{2} a = \dfrac{1 - \cos a}{1 + \cos a} = \dfrac{1 - \dfrac{\cos A}{\sin B}}{1 + \dfrac{\cos A}{\sin B}} = \dfrac{\sin B - \cos A}{\sin B + \cos A} = \dfrac{\sin B + \sin(A - 90°)}{\sin B - \sin(A - 90°)}$

$$= \frac{2 \sin \frac{1}{2}(A + B - 90°) \cos \frac{1}{2}(B - A + 90°)}{2 \cos \frac{1}{2}(A + B - 90°) \sin \frac{1}{2}(B - A + 90°)}$$
$$= \tan \tfrac{1}{2}(A + B - 90°) \cot \tfrac{1}{2}(B - A + 90°)$$
$$= \tan [\tfrac{1}{2}(A + B) - 45°] \cot [\tfrac{1}{2}(B - A) + 45°]$$
$$= \tan [\tfrac{1}{2}(A + B) - 45°] \tan [90° - \tfrac{1}{2}(B - A) - 45°]$$
$$= \tan [\tfrac{1}{2}(A + B) - 45°] \tan [\tfrac{1}{2}(A - B) + 45°].$$

6. $\tan^2(45° - \frac{1}{2}c) = \tan\frac{1}{2}(A - a)\cot\frac{1}{2}(A + a)$.

We have
$$\tan^2(45° - \tfrac{1}{2}c) = \frac{1 - \cos(90° - c)}{1 + \cos(90° - c)}$$
$$= \frac{1 - \dfrac{\sin a}{\sin A}}{1 + \dfrac{\sin a}{\sin A}} = \frac{\sin A - \sin a}{\sin A + \sin a}$$
$$= \frac{2\cos\frac{1}{2}(A + a)\sin\frac{1}{2}(A - a)}{2\sin\frac{1}{2}(A + a)\cos\frac{1}{2}(A - a)}$$
$$= \tan\tfrac{1}{2}(A - a)\cot\tfrac{1}{2}(A + a).$$

7. $\tan^2(45° - \frac{1}{2}b) = \dfrac{\sin(A - a)}{\sin(A + a)}$.

We have
$$\tan^2(45° - \tfrac{1}{2}b) = \frac{1 - \cos(90° - b)}{1 + \cos(90° - b)}$$
$$= \frac{1 - \dfrac{\tan a}{\tan A}}{1 + \dfrac{\tan a}{\tan A}} = \frac{\tan A - \tan a}{\tan A + \tan a}$$
$$= \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin a}{\cos a}}{\dfrac{\sin A}{\cos A} + \dfrac{\sin a}{\cos a}} = \frac{\sin A\cos a - \cos A\sin a}{\sin A\cos a + \cos A\sin a}$$
$$= \frac{\sin(A - a)}{\sin(A + a)}.$$

8. $\tan^2(45° - \frac{1}{2}B) = \tan\frac{1}{2}(A - a)\tan\frac{1}{2}(A + a)$.

The method of proof is similar to that given in the other cases.

Exercise 85. Formulas of Right Triangles

1. From the formula $\cos c = \cos a \cos b$ show that the hypotenuse of a right spherical triangle is less than 90° if the two sides are both less than 90° or are both greater than 90°.

2. As in Ex. 1, show that the hypotenuse is greater than 90° if one side is greater than 90° and the other side less than 90°.

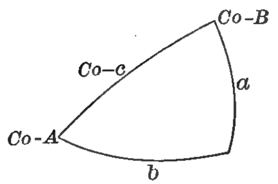
3. From the formula $\cos A = \cos a \sin B$ show that in a right spherical triangle an oblique angle and the opposite side are either both greater than 90° or both less than 90°.

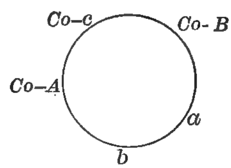
From the formulas on pages 193–195 state the inferences to be drawn respecting the values of the other parts when:

4. $c = 90°$. 6. $b = 90°$. 8. $a = b$. 10. $c = 90°$ and $a = 90°$.
5. $a = 90°$. 7. $c = a$. 9. $A = 90°$. 11. $a = 90°$ and $b = 90°$.

176. Napier's Rules. The ten formulas given on page 191 were very ingeniously reduced to two simple rules by John Napier, the inventor of logarithms. Since the right angle does not enter into the formulas, only five parts need be considered. Napier found that he could greatly simplify the treatment by considering:

1. The side a;
2. The side b;
3. The complement of A, called $Co\text{–}A$;
4. The complement of c, called $Co\text{–}c$;
5. The complement of B, called $Co\text{–}B$.





These parts are shown in the above triangle, C being omitted because it is not used. Since, as we shall see, it is convenient to consider any one of these as the *middle part* and the other parts as the *adjacent parts* and the *opposite parts*, they are often arranged on a circle as shown, and are known as *circular parts.*

If we speak of b as a middle part, $Co\text{–}A$ and a are the adjacent parts and $Co\text{–}c$ and $Co\text{–}B$ are the opposite parts.

The rules are as follows:

1. *The sine of any middle part is equal to the product of the tangents of the adjacent parts.*

2. *The sine of any middle part is equal to the product of the cosines of the opposite parts.*

These rules are easily remembered by the expressions *tan. ad.* and *cos. op.* While it is possible to get along very well without these rules, using the formulas on page 191, this is a convenient way of memorizing them.

177. Napier's Rules Verified. The correctness of Napier's rules may be easily shown by taking in turn each of the five parts as the middle part, and comparing with the formulas on page 191.

For example, let $Co\text{–}c$ be taken as the middle part; then $Co\text{–}A$ and $Co\text{–}B$ are the adjacent parts, and a and b the opposite parts, as is seen from the figure. Then, by Napier's rules,

$$\sin(Co\text{–}c) = \tan(Co\text{–}A)\tan(Co\text{–}B),$$

or

$$\cos c = \cot A \cot B;$$

and

$$\sin(Co\text{–}c) = \cos a \cos b,$$

or

$$\cos c = \cos a \cos b.$$

These results agree with formulas 10 and 1 on page 191.

Exercise 86. Spherical Triangles

Deduce eight of the formulas on page 191 by means of Napier's rules, taking for the middle part:

1. a. 2. b. 3. *Co–B*. 4. *Co–A*.

By Napier's rules deduce the following:

5. $\cos B = \tan a \cot c$. 6. $\sin a = \tan b \cot B$.

7. What do Napier's rules become if we take as the five parts of the triangle the hypotenuse, the two oblique angles, and the *complements* of the two sides?

8. Solve a spherical right triangle, given a, b, and c.

9. Solve a spherical right triangle, given A, B, and c.

10. Solve a spherical right triangle, given A, a, and b.

Find the number of degrees in the sides of a spherical triangle, given the angles of its polar triangle as follows:

11. $82°, 77°, 69°$.
12. $84\frac{1}{2}°, 81\frac{3}{4}°, 72\frac{1}{6}°$.
13. $78° 30', 89°, 102°$.
14. $83° 40', 48° 57', 103° 43'$.
15. $96° 37' 40'', 82° 29' 30'', 68° 47'$.
16. $43° 29' 37'', 98° 22' 53'', 87° 36' 39''$.

Find the number of degrees in the angles of a spherical triangle, given the sides of the polar triangle in Exs. 17–20:

17. $68° 42' 39'', 93° 48' 7'', 89° 38' 14''$.

18. $78° 47' 29'', 106° 36' 42''$, a quadrant.

19. $111° 29' 43''$, a quadrant, a quadrant.

20. A quadrant, half a quadrant, three fourths of a quadrant.

21. The angles of a spherical triangle are $70.5°$, $80.7°$, and $101.6°$. Find the sides of the polar triangle.

22. The sides of a spherical triangle are $40.72°$, $90°$, and $127.83°$. Find the angles of the polar triangle.

23. Show that, if a spherical triangle has three right angles, the sides of the triangle are quadrants.

24. Show that, if a spherical triangle has two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the opposite side.

25. How can the sides of a spherical triangle, measured in degrees, be found in units of length, when the length of the radius of the sphere is known?

178. Solution of the Right Spherical Triangle. By using either Napier's rules or the formulas on page 191, we can solve any right triangle if two parts besides the right angle are given.

It is a little easier to use the formulas, but the student who prefers to remember only Napier's rules can get on easily without charging his memory with the formulas or referring to page 191. The formulas given in the following solutions are all found on page 193.

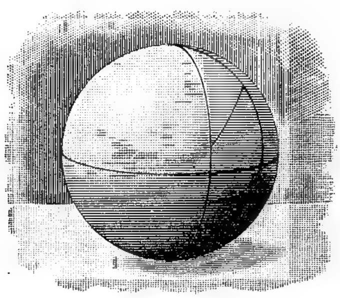
179. Given Two Sides. Given the two sides a and b of the right spherical triangle ACB, solve the triangle.

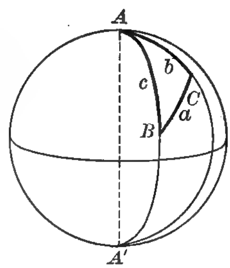
From $\cos c = \cos a \cos b$ we can find c;

then from $\tan A = \tan a \csc b$ we can find A;

and from $\tan B = \tan b \csc a$ we can find B.

For a check we can use $\cos c = \cot A \cot B$.





For example, in the right spherical triangle ACB, given $a = 27° 28' 36''$, $b = 51° 12' 8''$, solve the triangle.

$$\begin{aligned} \log \cos a &= 9.94802 \\ \log \cos b &= 9.79697 \\ \log \cos c &= 9.74499 \\ \therefore\ c &= 56° 13' 41'' \end{aligned} \qquad \begin{aligned} \log \tan a &= 9.71605 \\ \log \csc b &= 0.10826 \\ \log \tan A &= 9.82431 \\ \therefore\ A &= 33° 42' 51'' \end{aligned}$$

Check.

$$\begin{aligned} \log \tan b &= 10.09476 \\ \log \csc a &= 0.33594 \\ \log \tan B &= 10.43070 \\ \therefore\ B &= 69° 38' 54'' \end{aligned} \qquad \begin{aligned} \log \cot A &= 10.17569 \\ \log \cot B &= 9.56930 \\ \log \cos c &= 9.74499 \end{aligned}$$

If we know the diameter or the radius of the sphere, say in feet, we can find the circumference, and thus compute c in feet.

If c is very near 0° or 180°, it may be found to a greater degree of accuracy first by computing B from the formula $\tan B = \tan b \csc a$, and then computing c from the formula $\tan c = \tan a \sec B$.

Exercise 87. Given Two Sides

Solve the following right spherical triangles, given:

1. $a = 30°$, $b = 50°$.
2. $a = 40°$, $b = 60°$.
3. $a = 45°$, $b = 72°$.
4. $a = 56°$, $b = 78°$.
5. $a = 63°$, $b = 87°$.
6. $a = 68°$, $b = 93°$.
7. $a = 75°$, $b = 98°$.
8. $a = 82°$, $b = 100°$.
9. $a = 95°$, $b = 120°$.
10. $a = 120°$, $b = 119°$.
11. $a = 36° 27'$, $b = 43° 32' 31''$.
12. $a = 86° 40'$, $b = 32° 40'$.
13. $a = 50°$, $b = 36° 54' 49''$.
14. $a = 120° 10'$, $b = 150° 59' 44''$.
15. $a = 22° 15' 7''$, $b = 51° 53'$.
16. $a = 14° 16' 35''$, $b = 19° 17'$.
17. $a = 32° 9' 17''$, $b = 32° 41'$.
18. $a = 132° 14' 12''$, $b = 79° 13' 38''$.
19. $a = 2° 0' 55''$, $b = 0° 27' 10''$.
20. $a = 20° 20' 20''$, $b = 15° 16' 50''$.

21. How many degrees are there in the arc of a great circle drawn from a point on the equator in longitude 40° E. to a point on the prime meridian in latitude 40° N.?

22. Greenwich lies on the prime meridian 51° 28′ 38″ N. The arc of a great circle drawn from Greenwich to a point on the equator in longitude 25° W. makes what angle with the equator?

23. The arc of a great circle drawn from Greenwich to a point on the equator in longitude 150° E. makes what angle with the prime meridian?

24. How many degrees are there in the arc of a great circle drawn from a point on the equator in longitude 0° to a point in longitude 48° W., latitude 30° N.?

25. In a right spherical triangle on a sphere of radius 6 in. it is given that $a = 45°$ and $b = 70°$. Find the length of c in inches.

26. In a right spherical triangle on a sphere of diameter 2 ft. it is given that $a = 75°$ and $b = 75°$. Find the length of c in inches.

27. Taking the radius of the earth as 4000 mi., how many miles is it, on a great circle, from a point on the equator in longitude 70° W. to a point on the prime meridian in latitude 60° N.?

28. The arc of a great circle drawn from a point on the prime meridian 60° N. to a point on the equator 60° W. makes what angle with the prime meridian and with the equator?

29. In Ex. 28, what is the length of the arc, taking the radius of the earth as 4000 mi.?

180. Given the Hypotenuse and a Side. Given the hypotenuse c and the side a of the right spherical triangle ACB, solve the triangle.

From $\cos b = \cos c \sec a$ we can find b;

then from $\sin A = \sin a \csc c$ we can find A;

and from $\cos B = \tan a \cot c$ we can find B.

For a check we can use $\cos B = \cos b \sin A$.

Although two angles in general correspond to $\sin A$, one acute, the other obtuse, yet in this case it is easy to determine whether A is acute or obtuse, since A and a must both be greater than 90°, or both be less than 90°, as is apparent from the formula $\cos A = \cos a \sin B$, $\sin B$ always being positive in the spherical triangles considered, because B is less than 180°.

For a solution to be possible it is necessary and sufficient that $\sin a < \sin c$.

If b is very near 0° or 180°, it may be computed to a greater degree of accuracy by § 175, 1:

$$\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2}(c - a) \tan \tfrac{1}{2}(c + a).$$

If A is so near 90° that it cannot be found accurately in the tables, it may be computed from § 175, 2:

$$\tan^2 (45° - \tfrac{1}{2} A) = \tan \tfrac{1}{2}(c - a) \cot \tfrac{1}{2}(c + a).$$

If B cannot be found accurately, we may use § 175, 3, in this form:

$$\tan^2 \tfrac{1}{2} B = \sin (c - a) \csc (c + a).$$

Exercise 88. Given the Hypotenuse and a Side

Solve the right spherical triangles, given c and a as follows:

	c	a		c	a
1.	54° 20′	36° 27′	6.	44° 33′ 17″	32° 9′ 17″
2.	87° 11′ 40″	86° 40′	7.	97° 13′ 4″	132° 14′ 12″
3.	59° 4′ 26″	50°	8.	69° 25′ 11″	50°
4.	63° 55′ 43″	120° 10′	9.	2° 3′ 56″	2° 0′ 55″
5.	55° 9′ 32″	22° 15′ 7″	10.	90°	90°

11. A point on the equator in longitude 62° 30′ W. is 85° from a point A on the prime meridian. What is the latitude of A?

In a right spherical triangle show that:

12. $\cos^2 A \sin^2 c = \sin (c + a) \sin (c - a)$.

13. $\tan a \cos c = \sin b \cot B$.

14. If, in a right spherical triangle, p denotes the arc of the great circle passing through the vertex of the right angle and perpendicular to the hypotenuse, m and n the segments of the hypotenuse made by this arc adjacent to the sides a and b, show that $\tan^2 a = \tan c \tan m$, and that $\sin^2 p = \tan m \tan n$.

181. Given a Side and the Opposite Angle. Given the side a and the angle A of the right spherical triangle ACB, solve the angle.

From $\quad \sin c = \sin a \csc A$ we can find c;

then from $\quad \sin b = \tan a \cot A$ we can find b;

and from $\quad \sin B = \sec a \cos A$ we can find B.

Or we can find b and B from the formulas

$$\cos b = \cos c \sec a,$$

and

$$\cos B = \tan a \cot c.$$

For a check we can use $\sin b = \sin c \sin B$.

When c has been computed, b and B are determined by these values of their cosines; but since c must be found from its sine, c may have, in general, two values which are supplements of each other. This case, therefore, really admits of two solutions.

In fact, in the figure on page 198, if the sides b and c are extended until they meet in A', the two right triangles ABC and $A'BC$ have the side a in common, and $A = A'$. Also, $A'C = 180° - b$, $A'B = 180° - c$, and $\angle A'BC = 180° - B$. Hence, if ABC is one solution, $A'BC$ is the other.

For a solution to be possible it is necessary and sufficient that a and A shall both be greater or both less than 90° and that $\sin A > \sin a$.

When the formulas do not give accurate results, we may employ § 175, 6, 7, and 8:

$$\tan^2(45° - \tfrac{1}{2}c) = \tan\tfrac{1}{2}(A - a)\cot\tfrac{1}{2}(A + a),$$
$$\tan^2(45° - \tfrac{1}{2}b) = \sin(A - a)\csc(A + a),$$
$$\tan^2(45° - \tfrac{1}{2}B) = \tan\tfrac{1}{2}(A - a)\tan\tfrac{1}{2}(A + a).$$

Exercise 89. Given a Side and the Opposite Angle

Solve the right spherical triangles, given a and A as follows:

	a	A		a	A
1.	50°	63° 15′ 13″	7.	22° 15′ 7″	27° 28′ 38″
2.	36° 27′	46° 59′ 43″	8.	14° 16′ 35″	37° 36′ 49″
3.	86° 40′	88° 11′ 58″	9.	32° 9′ 17″	49° 20′ 16″
4.	120° 10′	105° 44′ 21″	10.	77° 21′ 50″	40° 40′ 40″
5.	115° 30′	110° 10′ 10″	11.	77° 21′ 50″	83° 56′ 40″
6.	122° 30′	120° 20′ 20″	12.	132° 14′ 12″	131° 43′ 50″

In a right spherical triangle show that:

13. $\sin^2 A = \cos^2 B + \sin^2 a \sin^2 B$.

14. $\sin(b + c) = 2\cos^2\tfrac{1}{2}A \cos b \sin c$.

15. $\sin(c - b) = 2\sin^2\tfrac{1}{2}A \cos b \sin c$.

182. Given a Side and an Adjacent Angle. Given the side a and the angle B of the right spherical triangle ACB, solve the triangle.

From $\tan c = \tan a \sec B$ we can find c;

then from $\tan b = \sin a \tan B$ we can find b;

and from $\cos A = \cos a \sin B$ we can find A.

For a check we can use $\cos A = \tan b \cot c$.

If A is near 0° or 180°, it may be found to a greater degree of accuracy by first computing b and then finding A from the formula $\tan A = \tan a \csc b$.

183. Given the Hypotenuse and an Angle. Given the hypotenuse c and the angle A of the right spherical triangle ACB, solve the triangle.

From $\sin a = \sin c \sin A$ we can find a;

then from $\tan b = \tan c \cos A$ we can find b;

and from $\cot B = \cos c \tan A$ we can find B.

For a check we can use $\sin a = \tan b \cot B$.

Here a is determined by $\sin a$, since a and A must both be greater than 90°, or both be less than 90°, as shown in § 180.

If a is near 90°, it may be found by first computing b and B, and then computing a by the formula $\sin a = \tan b \cot B$.

Exercise 90. Given a Side and an Adjacent Angle

Solve the right spherical triangles, given the following parts:

	a	B		c	A
1.	54° 30′	35° 30′	6.	91° 47′ 40″	92° 8′ 23″
2.	92° 47′ 32″	50° 2′ 1″	7.	25° 14′ 38″	54° 35′ 17″
3.	20° 20′ 20″	38° 10′ 10″	8.	59° 51′ 21″	70° 17′ 35″
4.	50°	63° 25′ 4″	9.	112° 48′	56° 11′ 56″
5.	50°	120° 3′ 50″	10.	2° 3′ 56″	77° 20′ 28″

11. Define a quadrantal triangle, and show how its solution may be reduced to that of the right triangle.

12. Solve the quadrantal triangle the sides of which are

$$a = 174° 12' 49'', \; b = 94° 8' 20'', \; c = 90°.$$

Solve the right spherical triangles, given the following parts:

13. $c = 55°$, $b = 45°$.
14. $c = 65°$, $A = 75°$.
15. $a = 110°$, $B = 45°$.
16. $A = 78°$, $c = 70°$.
17. $c = 50°$, $b = 44° 18' 39''$.
18. $A = 156° 20' 30''$, $a = 65° 15' 45''$.
19. $A = 74° 12' 31''$, $c = 64° 28' 47''$.
20. $a = 112° 42' 38''$, $B = 44° 28' 44''$.

184. Given the Two Angles. Given the angles A and B of the right spherical triangle ACB, solve the triangle.

From $\cos c = \cot A \cot B$ we can find c;

then from $\cos a = \cos A \csc B$ we can find a;

and from $\cos b = \cos B \csc A$ we can find b.

For a check we can use $\cos c = \cos a \cos b$.

For unfavorable values of the sides we can use formulas (§ 175):

$$\tan^2 \tfrac{1}{2} c = -\cos(A+B)\sec(A-B),$$

$$\tan^2 \tfrac{1}{2} a = \tan[\tfrac{1}{2}(A+B) - 45°]\tan[\tfrac{1}{2}(A-B) + 45°].$$

A solution is always possible if $A + B + C > 180°$, and if the difference between A and $B < 90°$.

185. Analogy to Plane Trigonometry. It is easy to trace analogies between the formulas for solving right spherical triangles and those for solving right plane triangles. The former become identical with the latter if we suppose the radius of the sphere to be infinite in length. Then the cosines of the sides become each equal to 1, and the ratios of the *sines* of the sides and of the *tangents* of the sides must be taken as equal to the ratios of the sides themselves.

186. Signs of the Functions. In solving spherical triangles write the algebraic sign of each function just above the function. Then the signs of the functions in the first members of equations like those of § 173 are + or − according as the law of signs makes the second members of the equations positive or negative.

If the function is a cosine, tangent, or cotangent, the + sign shows the angle $< 90°$, the − sign shows the angle $> 90°$, and then the *supplement* of the angle obtained from the table must be taken.

If the function is a sine, the acute angle obtained from the table and the supplement of this angle must be considered as solutions unless there are other conditions that remove the ambiguity.

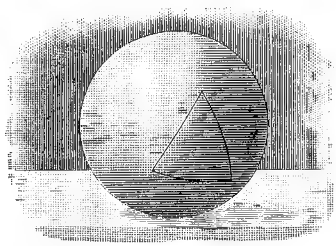
Exercise 91. Given the Two Angles

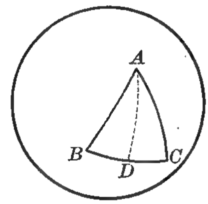
Solve the right spherical triangles, given A and B as follows:

	A	B		A	B
1.	63° 15′ 12″	135° 33′ 39″	6.	77° 20′ 28″	12° 40′
2.	116° 43′ 12″	116° 31′ 25″	7.	54° 35′ 17″	38° 10′ 10″
3.	46° 59′ 43″	57° 59′ 19″	8.	70° 17′ 35″	35° 30′
4.	90°	88° 24′ 35″	9.	54° 54′ 42″	63° 25′ 4″
5.	92° 8′ 23″	50° 2′ 1″	10.	56° 11′ 56″	120° 3′ 50″

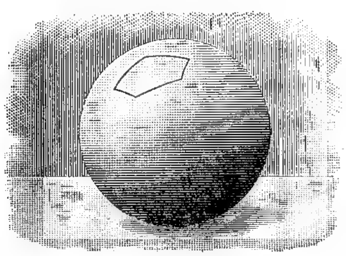
187. The Isosceles Spherical Triangle. The solution of an isosceles spherical triangle may be reduced to that of a right spherical triangle.

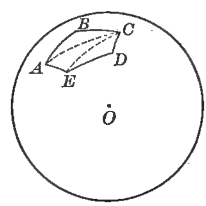
For an arc of the great circle passed through the vertex of an isosceles spherical triangle and the mid-point of the base divides the triangle into two equivalent right spherical triangles.





188. The Regular Spherical Polygon. A spherical polygon formed by the intersections of the spherical surface with the faces of a regular pyramid whose vertex is at the center of the sphere is called a *regular spherical polygon.*





The solution of a regular spherical polygon may be reduced to that of a right spherical triangle.

For arcs of great circles through the center of the polygon and the vertices divide the polygon into congruent isosceles triangles which can be solved (§ 187).

Exercise 92. Isosceles Triangles

Solve the isosceles spherical triangles, given:

1. $c = 50°$, $a = 30°$.
2. $c = 60°$, $a = 40°$.
3. $c = 62° 37'$, $a = 49° 10'$.
4. $c = 29° 35'$, $B = 15°$.
5. $c = 68° 47'$, $B = 42° 30'$.
6. $c = 79° 49'$, $B = 49° 37'$.

7. In an isosceles spherical triangle, given the base a and the side b, find B, A, and AD, as shown in the above figure.

CHAPTER II

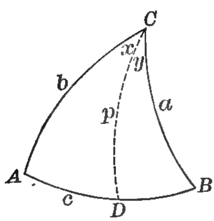
THE OBLIQUE SPHERICAL TRIANGLE

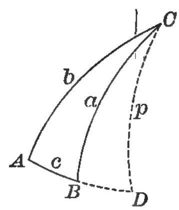
189. Law of Sines. In the oblique spherical triangle ABC let p be the perpendicular from C to AB, as shown. Then in either figure, from § 172, 3,

$$\sin p = \sin a \sin B,$$

and

$$\sin p = \sin b \sin A.$$





Dividing, we have
$$1 = \frac{\sin a}{\sin A} \cdot \frac{\sin B}{\sin b},$$

or
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

Similarly,
$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Hence
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

That is, in any spherical triangle,

The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.

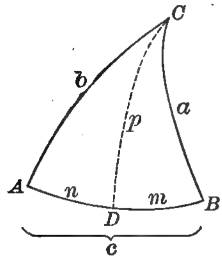
Exercise 93. Law of Sines

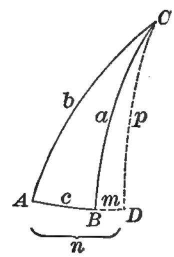
Consider the Law of Sines when:

1. $A = 90°$.
2. $B = 90°$.
3. $C = 90°$.
4. $a = 90°$.
5. $A = B = 90°$.
6. $a = b = 90°$.
7. $a = A = 90°$.
8. $a = b = A = B = 90°$.
9. $A = B = C = 90°$.

190. Law of Cosines of Sides. Drawing the figures as in § 189 we see, from § 172, that

$$\cos a = \cos p \cos m = \cos p \cos (c \sim n)$$
$$= \cos p \cos c \cos n + \cos p \sin c \sin n.$$





Furthermore, in the right spherical triangle ADC, from § 172,

$$\cos p \cos n = \cos b,$$

whence $$\cos p = \cos b \sec n,$$

and $$\cos p \sin n = \cos b \tan n.$$

And since $$\tan n = \tan b \cos A, \text{ by § 174},$$

we have $$\cos p \sin n = \cos b \tan b \cos A$$
$$= \sin b \cos A.$$

Substituting in the value of $\cos a$, we have

$$\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos A;}$$

and, similarly, $$\mathbf{\cos b = \cos c \cos a + \sin c \sin a \cos B,}$$
$$\mathbf{\cos c = \cos a \cos b + \sin a \sin b \cos C.}$$

Exercise 94. Law of Cosines of Sides

Consider the Law of Cosines of Sides when:

1. $A = 90°$. 2. $B = 90°$. 3. $A = B = 90°$. 4. $A = B = C = 90°$.

Prove the following formulas:

5. $1 - \cos a = 1 - \cos (b - c) + \sin b \sin c \operatorname{versin} A$.

6. $\operatorname{versin} a = \operatorname{versin} (b - c) \left[1 + \dfrac{\sin b \sin c \operatorname{versin} A}{\operatorname{versin} (b - c)}\right]$.

7. From the Law of Cosines find formulas for $\cos A$, $\cos B$, and $\cos C$ in terms of functions of a, b, and c.

8. Prove that $\cos c = \dfrac{\cos a - \sin b \sin c \cos A}{\cos b}$.

9. In the figures given above prove that $\cos p = \cos a \sec m$.

191. Law of Cosines of Angles. From this figure, or from the second figure on page 205, we have (§ 172)

$$\begin{aligned}\cos A &= \cos p \sin x\\ &= \cos p \sin (C - y)\\ &= \cos p \sin C \cos y - \cos p \cos C \sin y.\end{aligned}$$

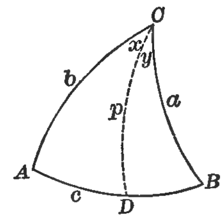
Furthermore, by § 172,

$$\cos p \sin y = \cos B.$$

Therefore $\qquad \cos p = \cos B \csc y,$

and

$$\begin{aligned}\cos p \cos y &= \cos B \cot y\\ &= \cos B \tan B \cos a\\ &= \sin B \cos a.\end{aligned}$$



Substituting these values of $\cos p \sin y$ and $\cos p \cos y$ in the value of $\cos A$, we obtain

$$\boldsymbol{\cos A = -\cos B \cos C + \sin B \sin C \cos a;}$$

and, similarly,

$$\boldsymbol{\cos B = -\cos A \cos C + \sin A \sin C \cos b,}$$

$$\boldsymbol{\cos C = -\cos A \cos B + \sin A \sin B \cos c.}$$

It will be observed that the formulas for $\cos A$, $\cos B$, and $\cos C$ are derived from those for $\cos a$, $\cos b$, and $\cos c$ by interchanging capital and small letters, and changing the sign of one product. In general, it is easily shown that each part of a spherical triangle may be replaced by the supplement of the opposite part, and this is the Principle of Duality of spherical triangles.

Exercise 95. Law of Cosines of Angles

Consider the Law of Cosines of Angles when:

1. $A = 0°$. 2. $A = 180°$. 3. $A = 90°$. 4. $A = B = 90°$.

5. Deduce the formulas of § 191 from those of § 190 by means of the relations between polar triangles (§ 171).

Prove the following formulas:

6. $1 - \cos A = 1 - \cos (B - C) + \sin B \sin C \operatorname{versin} a.$

7. $\operatorname{versin} A = \operatorname{versin} (B - C)\left[1 + \dfrac{\sin B \sin C \operatorname{versin} a}{\operatorname{versin} (B - C)}\right].$

From the Law of Cosines find formulas for the following in terms of functions of A, B, and C:

8. $\cos a$. 9. $\cos b$. 10. $\cos c$.

11. Investigate the dual of Ex. 8 in Exercise 94.

192. Formulas for Half Angles. Since we have, from the Law of Cosines of Sides (§ 190),

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

we see that
$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence
$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c} \qquad \text{§ 96}$$

$$= \frac{-2 \sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(b - c - a)}{\sin b \sin c}. \qquad \text{§ 103}$$

Similarly,
$$1 + \cos A = \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c} \qquad \text{§ 91}$$

$$= \frac{-2 \sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(a - b - c)}{\sin b \sin c}. \qquad \text{§ 103}$$

But it was shown in § 102 that

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A.$$

$$\therefore \sin^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(a - b + c)}{\sin b \sin c}.$$

It was also shown in § 102 that

$$1 + \cos A = 2 \cos^2 \tfrac{1}{2} A. \qquad \text{§ 102}$$

$$\therefore \cos^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(b + c - a)}{\sin b \sin c}.$$

Let s represent the semiperimeter of the triangle; that is, let

$$\tfrac{1}{2}(a + b + c) = s.$$

Then
$$\tfrac{1}{2}(b + c - a) = s - a,$$
$$\tfrac{1}{2}(a - b + c) = s - b,$$
and
$$\tfrac{1}{2}(a + b - c) = s - c.$$

Substituting these values in the above formulas, and extracting the square roots, we have

$$\boldsymbol{\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}},}$$

$$\boldsymbol{\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.}$$

Dividing,
$$\boldsymbol{\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}.}$$

In like manner, the following formulas can be proved:

For angle B,
$$\sin\tfrac{1}{2}B=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}},$$
$$\cos\tfrac{1}{2}B=\sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}},$$
$$\tan\tfrac{1}{2}B=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}};$$

For angle C,
$$\sin\tfrac{1}{2}C=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}},$$
$$\cos\tfrac{1}{2}C=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}},$$
$$\tan\tfrac{1}{2}C=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}.$$

Exercise 96. Formulas for Half Angles

Show that the following formulas are true:

1. $\sin\frac{1}{2}A=\sqrt{\sin(s-b)\sin(s-c)\csc b\csc c}.$
2. $\cos\frac{1}{2}A=\sqrt{\sin s\sin(s-a)\csc b\csc c}.$

Find the value of A in each case, given:

3. $a=95°,\ b=58°,\ c=42°.$
4. $a=92°,\ b=61°,\ c=43°.$
5. $a=96°,\ b=64°,\ c=48°.$
6. $a=98°,\ b=78°,\ c=60°.$

Find the value of B in each case, given:

7. $a=95°,\ b=60°,\ c=40°.$
8. $a=97°,\ b=62°,\ c=38°.$

Find the value of C in each case, given:

9. $a=92°,\ b=59°,\ c=37°.$
10. $a=96°,\ b=64°,\ c=39°.$

Prove the following formulas:

11. $\sin\frac{1}{2}(180°-A)=\sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}}.$

12. $\cos\frac{1}{2}(180°-A)=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$

13. $\tan\frac{1}{2}(180°-A)=\sqrt{\dfrac{\sin s\sin(s-a)}{\sin(s-b)\sin(s-c)}}.$

193. Formulas for Half Sides. Since, by the Law of Cosines of Angles (§ 91),

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

we have
$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}.$$

$$\therefore 1 - \cos a = \frac{\sin B \sin C - \cos B \cos C - \cos A}{\sin B \sin C}$$

$$= \frac{-\cos(B + C) - \cos A}{\sin B \sin C} \qquad \S\ 91$$

$$= \frac{-2\cos\frac{1}{2}(B + C + A)\cos\frac{1}{2}(B + C - A)}{\sin B \sin C}. \qquad \S\ 103$$

Also,
$$1 + \cos a = \frac{\sin B \sin C + \cos B \cos C + \cos A}{\sin B \sin C}$$

$$= \frac{\cos(B - C) + \cos A}{\sin B \sin C} \qquad \S\ 96$$

$$= \frac{2\cos\frac{1}{2}(B - C + A)\cos\frac{1}{2}(B - C - A)}{\sin B \sin C}. \qquad \S\ 103$$

But it was shown in § 102 that

$$1 - \cos a = 2\sin^2\tfrac{1}{2}a.$$

$$\therefore \sin^2\tfrac{1}{2}a = \frac{-\cos\frac{1}{2}(B + C + A)\cos\frac{1}{2}(B + C - A)}{\sin B \sin C}.$$

It was also shown in § 102 that

$$1 + \cos a = 2\cos^2\tfrac{1}{2}a.$$

$$\therefore \cos^2\tfrac{1}{2}a = \frac{\cos\frac{1}{2}(B - C + A)\cos\frac{1}{2}(B - C - A)}{\sin B \sin C}.$$

Now let $\quad \frac{1}{2}(A + B + C) = S.$

Then $\quad \frac{1}{2}(B + C - A) = S - A,$

$\quad \frac{1}{2}(A - B + C) = S - B,$

and $\quad \frac{1}{2}(A + B - C) = S - C.$

Substituting these values in the above formulas and extracting the square roots, we have

$$\sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S - A)}{\sin B \sin C}},$$

$$\cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S - B)\cos(S - C)}{\sin B \sin C}}.$$

Dividing,
$$\tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos(S - A)}{\cos(S - B)\cos(S - C)}}.$$

In like manner, writing b, c, a, for a, b, c respectively, and B, C, A, for A, B, C respectively, we have the following formulas:

For side b,
$$\sin\tfrac{1}{2}b=\sqrt{\frac{-\cos S\cos(S-B)}{\sin A\sin C}},$$
$$\cos\tfrac{1}{2}b=\sqrt{\frac{\cos(S-A)\cos(S-C)}{\sin A\sin C}},$$
$$\tan\tfrac{1}{2}b=\sqrt{\frac{-\cos S\cos(S-B)}{\cos(S-A)\cos(S-C)}};$$

For side c,
$$\sin\tfrac{1}{2}c=\sqrt{\frac{-\cos S\cos(S-C)}{\sin A\sin B}},$$
$$\cos\tfrac{1}{2}c=\sqrt{\frac{\cos(S-A)\cos(S-B)}{\sin A\sin B}},$$
$$\tan\tfrac{1}{2}c=\sqrt{\frac{-\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}}.$$

Exercise 97. Formulas for Half Sides

Consider the formula for $\sin\frac{1}{2}a$ *when:*

1. $B=90°$. 2. $C=90°$. 3. $B=C=90°$.

Consider the formula for $\sin\frac{1}{2}b$ *when:*

4. $A=45°$. 5. $C=45°$. 6. $A=C=45°$.

Consider the formula for $\sin\frac{1}{2}c$ *when:*

7. $A=200°$, $B=100°$, $C=135°$. 8. $A=B=C=90°$.

Show that the following formulas are true:

9. $\sin\frac{1}{2}a=\sqrt{-\cos S\cos(S-A)\csc B\csc C}$.

10. $\cos\frac{1}{2}a=\sqrt{\cos(S-B)\cos(S-C)\csc B\csc C}$.

11. $\tan\frac{1}{2}a=\sqrt{-\cos S\cos(S-A)\sec(S-B)\sec(S-C)}$.

12. $\sin\frac{1}{2}b=\sqrt{-\cos S\cos(S-B)\csc A\csc C}$.

13. $\tan\frac{1}{2}c=\sqrt{-\cos S\cos(S-C)\sec(S-A)\sec(S-B)}$.

14. From the formula for $\tan\frac{1}{2}b$ deduce another formula similar to that of Ex. 13.

15. From the formula for $\cos\frac{1}{2}b$ deduce another formula similar to that of Ex. 10.

16. From the formula for $\sin\frac{1}{2}c$ deduce another formula similar to that of Ex. 9.

194. Gauss's Equations. From § 91 we have

$$\cos \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B - \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B.$$

Substituting the values found in § 192 we have

$$\cos \tfrac{1}{2}(A+B) = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \times \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}}$$

$$- \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \times \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}}$$

$$= \frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}$$

$$- \frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}$$

$$= \frac{\sin s - \sin(s-c)}{\sin c} \times \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}.$$

By § 103, $\sin s - \sin(s-c) = 2\cos \tfrac{1}{2}(s+s-c)\sin \tfrac{1}{2}(s-s+c)$
$= 2\cos(s - \tfrac{1}{2}c)\sin \tfrac{1}{2}c;$

by § 101, $\sin c = 2\sin \tfrac{1}{2}c \cos \tfrac{1}{2}c;$

and by § 192, $\sqrt{\dfrac{\sin(s-a)\sin(s-b)}{\sin a \sin b}} = \sin \tfrac{1}{2}C.$

Substituting in the value of $\cos \tfrac{1}{2}(A+B)$, we have

$$\cos \tfrac{1}{2}(A+B) = \frac{2\cos(s-\tfrac{1}{2}c)\sin \tfrac{1}{2}c}{2\sin \tfrac{1}{2}c \cos \tfrac{1}{2}c}\sin \tfrac{1}{2}C$$

$$= \frac{\cos(s-\tfrac{1}{2}c)}{\cos \tfrac{1}{2}c}\sin \tfrac{1}{2}C.$$

$$\therefore \cos \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}c = \cos(s-\tfrac{1}{2}c)\sin \tfrac{1}{2}C.$$

But $s - \tfrac{1}{2}c = \tfrac{1}{2}(a+b).$

$$\therefore \cos \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a+b)\sin \tfrac{1}{2}C.$$

By proceeding in like manner with the values of

$$\sin \tfrac{1}{2}(A+B),\ \cos \tfrac{1}{2}(A-B),\text{ and } \sin \tfrac{1}{2}(A-B),$$

three analogous equations are obtained.

The four equations,

$$\boldsymbol{\cos \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a+b)\sin \tfrac{1}{2}C,}$$
$$\boldsymbol{\sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}c = \cos \tfrac{1}{2}(a-b)\cos \tfrac{1}{2}C,}$$
$$\boldsymbol{\cos \tfrac{1}{2}(A-B)\sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a+b)\sin \tfrac{1}{2}C,}$$
$$\boldsymbol{\sin \tfrac{1}{2}(A-B)\sin \tfrac{1}{2}c = \sin \tfrac{1}{2}(a-b)\cos \tfrac{1}{2}C,}$$

are called Gauss's Equations from the great German mathematician.

195. Napier's Analogies. By dividing the second of Gauss's Equations by the first, the fourth by the third, the third by the first, and the fourth by the second, we obtain

$$\tan\frac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C,$$

$$\tan\frac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}C,$$

$$\tan\frac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c,$$

$$\tan\frac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c.$$

There will be other forms in each case, according as other elements of the triangle are used.

Although these equations are not identical with those of plane trigonometry, as given in §§ 103, 112, they are analogous to them. For example, from § 103 we can derive

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin A-\sin B}{\sin A+\sin B}\cot\tfrac{1}{2}C,$$

which is analogous to the above formula. These relations are known as Napier's Analogies, having been discovered by Napier, the inventor of logarithms.

In the first equation the factors $\cos\frac{1}{2}(a-b)$ and $\cot\frac{1}{2}C$ are always positive; therefore $\tan\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ must always have like signs.

Hence, if $a+b<180°$, then $\cos\frac{1}{2}(a+b)>0$ and $\tan\frac{1}{2}(A+B)>0$. Hence $A+B<180°$.

If $a+b>180°$, then $A+B>180°$.

If $a+b=180°$, $\cos\frac{1}{2}(a+b)=0$ and $\tan\frac{1}{2}(A+B)=\infty$. Hence $\frac{1}{2}(A+B)=90°$, and $A+B=180°$.

Conversely, it may be shown from the third equation that $a+b$ is less than, greater than, or equal to 180° according as $A+B$ is less than, greater than, or equal to 180°. That is,

In a spherical triangle the sum of any two sides is less than, greater than, or equal to 180° according as the sum of their opposite angles is less than, greater than, or equal to 180°.

196. Solution of the Oblique Spherical Triangle. By using either Gauss's Equations or Napier's Analogies we can solve any oblique spherical triangle if three parts are known.

In certain cases, however, more than one solution is possible, as is also true in plane trigonometry. These cases will be discussed when they arise.

197. Given Two Sides and the Included Angle. For example, given a, b, and C, solve the triangle.

The angles A and B may be found by the first two of Napier's Analogies:

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C;$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

After A and B have been found, the side c can be found by § 189 or by § 193; but it is better to use for this purpose Gauss's Equations, because they involve the functions of the same angles that occur in working Napier's Analogies. Any one of the equations may be used; for example,

$$\cos\tfrac{1}{2}c=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(A+B)}\sin\tfrac{1}{2}C.$$

For example, given $a=73°\,58'\,54''$, $b=38°\,45'$, $C=46°\,33'\,41''$, solve the triangle.

$$a=73°\,58'\,54'' \qquad \therefore \tfrac{1}{2}(a-b)=17°\,36'\,57''$$
$$b=38°\,45'\,0'' \qquad \tfrac{1}{2}(a+b)=56°\,21'\,57''$$
$$C=46°\,33'\,41'' \qquad \tfrac{1}{2}C=23°\,16'\,50.5''$$

$$\begin{aligned}
\log\cos\tfrac{1}{2}(a-b)&=9.97914\\
\text{colog}\cos\tfrac{1}{2}(a+b)&=0.25658\\
\log\cot\tfrac{1}{2}C&=\underline{0.36626}\\
\log\tan\tfrac{1}{2}(A+B)&=\underline{0.60198}\\
\log\cos\tfrac{1}{2}(a+b)&=9.74342\\
\text{colog}\cos\tfrac{1}{2}(A+B)&=0.61515\\
\log\sin\tfrac{1}{2}C&=\underline{9.59685}\\
\log\cos\tfrac{1}{2}c&=9.95542\\
\therefore \tfrac{1}{2}c&=25°\,31'\,10''
\end{aligned}
\qquad
\begin{aligned}
\log\sin\tfrac{1}{2}(a-b)&=9.48092\\
\text{colog}\sin\tfrac{1}{2}(a+b)&=0.07956\\
\log\cot\tfrac{1}{2}C&=\underline{0.36626}\\
\log\tan\tfrac{1}{2}(A-B)&=\underline{9.92674}\\
\therefore \tfrac{1}{2}(A+B)&=75°\,57'\,40.8''\\
\therefore \tfrac{1}{2}(A-B)&=40°\,11'\,25.4''\\
&\left\{\begin{aligned}A&=116°\;\;9'\;\;6''\\ B&=35°\,46'\,15''\\ c&=51°\;\;2'\,20''\end{aligned}\right.
\end{aligned}$$

To test the accuracy of the work we may use the Law of Sines (§ 189).

Exercise 98. Given Two Sides and the Included Angle

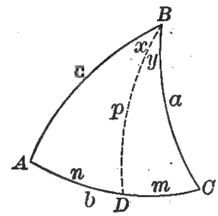
Solve the triangles, given the following parts:

1. $a=88°\,12'\,20''$, $b=124°\,7'\,17''$, $C=50°\,2'\,1''$.
2. $a=120°\,55'\,35''$, $b=88°\,12'\,20''$, $C=47°\,42'\,1''$.
3. $b=63°\,15'\,12''$, $c=47°\,42'\,1''$, $A=59°\,4'\,25''$.
4. $b=69°\,25'\,11''$, $c=109°\,46'\,19''$, $A=54°\,54'\,42''$.
5. Two sides of a triangle are 90° and 12°, and the included angle is 85°. Find the third side in degrees.

198. To find the Third Side. As a special case of § 197 we occasionally have given two sides and the included angle, to find only the third side; that is, to find c without previously computing A and B. For this purpose we might use the Law of Cosines (§ 190),

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

But this is not adapted to work with logarithms, and hence we employ a method used in the study of the right triangle.



In the figure let BD be perpendicular to AC, and then letter the parts as shown. We then have

$$\cos C = \tan m \cot a,$$

whence $$\tan m = \tan a \cos C.$$

Furthermore, by § 172,

$$\cos a = \cos m \cos p, \text{ whence } \cos p = \cos a \sec m,$$

and $$\cos c = \cos n \cos p, \text{ whence } \cos p = \cos c \sec n.$$

Therefore $$\cos c \sec n = \cos a \sec m.$$

Since $$n = b - m,$$

$$\cos c = \cos a \sec m \cos (b - m).$$

Now c may be computed from the two equations

$$\tan m = \tan a \cos C,$$

and $$\cos c = \cos a \sec m \cos (b - m).$$

If BD falls without the triangle, for instance to the right of BC, then $n = b + m$.

$$\therefore \cos c = \cos a \sec m \cos (b + m).$$

For example, given $a = 97° 30'$, $b = 55° 12'$, $C = 39° 58'$, find c. Writing (n) to indicate a negative function, we have

$\log \tan a = 0.88057\ (n)$	$\log \cos a = 9.11570\ (n)$
$\log \cos C = 9.88447$	$\log \sec m = 0.77135\ (n)$
$\log \tan m = 0.76504\ (n)$	$\log \cos (b - m) = 9.85289$
$\therefore m = 99° 44' 49''$	$\log \cos c = 9.73994$
$\therefore b - m = -44° 32' 49''$	$\therefore c = 56° 40' 9''$

Exercise 99. To find the Third Side

Find the value of c, given the following parts:

1. $a = 88° 30'$, $b = 125° 45'$, $C = 49° 15'$.
2. $a = 121° 45'$, $b = 92° 15'$, $C = 48° 30'$.
3. $a = 63.5°$, $b = 89.25°$, $C = 52.75°$.
4. $a = 72.25°$, $b = 93.75°$, $C = 63.5°$.

199. Given Two Angles and the Included Side. For example, given A, B, and c. The sides a and b can be found from the formulas

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c,$$

and

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c. \qquad \text{§ 195}$$

The angle C can then be found by the formulas of § 189, § 194, or § 195. Thus, from § 194 we have

$$\cos \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(a-b)} \cos \tfrac{1}{2} c.$$

For example, given $A = 107° 47' 7''$, $B = 38° 58' 27''$, $c = 51° 41' 14''$, solve the triangle.

$$\begin{aligned} A &= 107° 47' \ \ 7'' \\ B &= \ \ 38° 58' 27'' \\ c &= \ \ 51° 41' 14'' \end{aligned} \qquad \begin{aligned} \therefore \tfrac{1}{2}(A-B) &= 34° 24' 20'' \\ \tfrac{1}{2}(A+B) &= 73° 22' 47'' \\ \tfrac{1}{2} c &= 25° 50' 37'' \end{aligned}$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(A-B) &= 9.91648 \\ \operatorname{colog} \cos \tfrac{1}{2}(A+B) &= 0.54359 \\ \log \tan \tfrac{1}{2} c &= 9.68517 \\ \log \tan \tfrac{1}{2}(a+b) &= 0.14524 \\ \log \sin \tfrac{1}{2}(A+B) &= 9.98146 \\ \operatorname{colog} \cos \tfrac{1}{2}(a-b) &= 0.01703 \\ \log \cos \tfrac{1}{2} c &= 9.95423 \\ \log \cos \tfrac{1}{2} C &= 9.95272 \\ \therefore \tfrac{1}{2} C &= 26° 15' 10'' \end{aligned} \qquad \begin{aligned} \log \sin \tfrac{1}{2}(A-B) &= 9.75208 \\ \operatorname{colog} \sin \tfrac{1}{2}(A+B) &= 0.01854 \\ \log \tan \tfrac{1}{2} c &= 9.68517 \\ \log \tan \tfrac{1}{2}(a-b) &= 9.45579 \\ \therefore \tfrac{1}{2}(a+b) &= 54° 24' 24.4'' \\ \therefore \tfrac{1}{2}(a-b) &= 15° 56' 25.5'' \\ &\left\{\begin{aligned} a &= 70° 20' 50'' \\ b &= 38° 27' 59'' \\ C &= 52° 30' 20'' \end{aligned}\right. \end{aligned}$$

Exercise 100. Given Two Angles and the Included Side

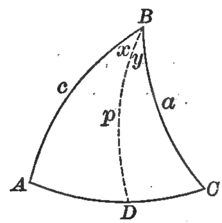
1. Write the formulas used in computing A, given B, C, and a.
2. Write the formulas used in computing B, given A, C, and b.
3. Write the formulas used in computing b, given B, C, and a.

Solve the triangles, given the following parts:

4. $A = 28°$, $B = 40°$, $c = 90°$.
5. $A = 35°$, $B = 56°$, $c = 70°$.
6. $A = 46°$, $B = 60°$, $c = 80°$.
7. $A = 75°$, $B = 30°$, $c = 85°$.
8. $A = 60°$, $B = 60°$, $c = 40°$.
9. $A = 80°$, $B = 80°$, $c = 80°$.
10. $A = 26°$, $B = 39°$, $c = 154°$.
11. $A = 128°$, $B = 107°$, $c = 124°$.
12. $A = 153°$, $C = 78°$, $b = 86°$.
13. $A = 125°$, $C = 82°$, $b = 52°$.
14. $A = 100°$, $C = 90°$, $b = 72°$.
15. $A = 120°$, $C = 88°$, $b = 75°$.

200. To find the Third Angle. As a special case of § 199 we may have given two angles A and B and the included side c, to find only the third angle, C.

This is analogous to the case given in § 198, and we proceed in the same manner, dividing the triangle into right triangles by drawing BD perpendicular to AC, and lettering the figure as here shown.



Since, by § 172, $\cos c = \cot x \cot A,$

we have $\cot x = \tan A \cos c.$

Since $\cos A = \cos p \sin x,$

we have $$\cos p = \frac{\cos A}{\sin x}.$$

Since $\cos C = \cos p \sin y,$

we have $$\cos p = \frac{\cos C}{\sin y},$$

and $$\cos C = \frac{\cos A \sin y}{\sin x} = \frac{\cos A \sin (B - x)}{\sin x}.$$

Hence C can be computed from the two equations

$$\cot x = \tan A \cos c,$$

$$\cos C = \frac{\cos A \sin (B - x)}{\sin x}.$$

When BD falls to the right of BC the last equation becomes

$$\cos C = \cos A \sin (x - B) \sin x.$$

For example, given $A = 35° 46' 14''$, $B = 115° 9' 7''$, $c = 51° 2' 30''$, find C.

$$\begin{aligned} \log \tan A &= 9.85760 \\ \log \cos c &= \underline{9.79848} \\ \log \cot x &= 9.65608 \\ \therefore x &= 65° 37' 49'' \\ \therefore B - x &= 49° 31' 18'' \end{aligned} \qquad \begin{aligned} \log \cos A &= 9.90922 \\ \log \sin (B - x) &= 9.88118 \\ \text{colog} \sin x &= \underline{0.04053} \\ \log \cos C &= 9.83093 \\ \therefore C &= 47° 20' 56'' \end{aligned}$$

Exercise 101. To find the Third Angle

Find the value of C, given the following parts:

1. $A = 28°$, $B = 40°$, $c = 120°$.
2. $A = 35°$, $B = 45°$, $c = 130°$.
3. $A = 120°$, $B = 100°$, $c = 130°$.
4. $A = 140°$, $B = 75°$, $c = 125°$.

Find the value of the third angle, given the following parts:

5. $A = 26° 58' 46''$, $B = 39° 45' 10''$, $c = 154° 46' 48''$.
6. $A = 128° 41' 49''$, $B = 107° 33' 20''$, $c = 124° 12' 31''$.

201. Given Two Sides and an Angle opposite one of them. For example, given a, b, and A, solve the triangle.

As in Plane Trigonometry (§§ 108, 109), this results in more than one solution in certain cases considered below.

From the Law of Sines (§ 189),

$$\sin B = \frac{\sin A \sin b}{\sin a},$$

whence B can be found, a, b, and A being given.

We may now find C and c from the formulas of § 195, written thus:

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b),$$

and

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B).$$

Since B is determined from its sine, the problem in general has two solutions; and, moreover, in case $\sin B > 1$, the problem is impossible. By geometric construction it may be shown, as in the corresponding case in Plane Trigonometry (§§ 108, 109), under what conditions the problem really has two solutions, one solution, or no solution. But in practical applications a general knowledge of the shape of the triangle is known beforehand, so that it is easy to see, without special investigation, which solution (if any) corresponds to the circumstances of the question.

It can be shown that there are *two solutions* when A and a are alike in kind and $\sin b > \sin a > \sin A \sin b$; *no solution* when A and a are unlike in kind (including the case in which either A or a is 90°) and $\sin b > \sin a$ or $\sin b = \sin a$, or when $\sin a < \sin A \sin b$; and *one solution* in every other case.

The side c or the angle C may be computed, without first finding B, by means of the formulas

$$\tan m = \cos A \tan b, \text{ and } \cos(c-m) = \cos a \sec b \cos m;$$
$$\cot x = \tan A \cos b, \text{ and } \cos(C-x) = \cot a \tan b \cos x.$$

These formulas may be obtained by resolving the triangle into right triangles, and then applying Napier's Rules; m is equal to that part of the side c included between the vertex A and the foot of the perpendicular from C, and x is equal to the corresponding portion of the angle C.

For example, given $a = 57° 36'$, $b = 31° 14'$, $A = 104° 25' 30''$.

In this case	$A > 90°$,	$\log \sin A =$	9.98609
and	$a + b < 180°$.	$\log \sin b =$	9.71477
Therefore	$A + B < 180°$,	$\text{colog} \sin a =$	0.07349
and	$B < 90°$.	$\log \sin B =$	9.77435
Hence there is only one solution.		$\therefore B =$	$36° 29' 46''$

Having now found B, we can proceed by the formulas given above to find c and C.

We first use the formula for $\tan \frac{1}{2} c$, and then the formula for $\tan \frac{1}{2} C$, as given on page 218, thus:

$$\begin{aligned} a+b &= 88^\circ\ 50' & \tfrac{1}{2}(a+b) &= 44^\circ\ 25' \\ a-b &= 26^\circ\ 22' & \tfrac{1}{2}(a-b) &= 13^\circ\ 11' \\ A+B &= 140^\circ\ 55'\ 16'' & \tfrac{1}{2}(A+B) &= 70^\circ\ 27'\ 38'' \\ A-B &= \ \ 67^\circ\ 55'\ 44'' & \tfrac{1}{2}(A-B) &= 33^\circ\ 57'\ 52'' \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(A+B) &= 9.97424 & \log \sin \tfrac{1}{2}(a+b) &= 9.84502 \\ \text{colog} \sin \tfrac{1}{2}(A-B) &= 0.25284 & \text{colog} \sin \tfrac{1}{2}(a-b) &= 0.64194 \\ \log \tan \tfrac{1}{2}(a-b) &= \underline{9.36966} & \log \tan \tfrac{1}{2}(A-B) &= \underline{9.82840} \\ \log \tan \tfrac{1}{2} c &= 9.59674 & \log \cot \tfrac{1}{2} C &= 0.31536 \\ \therefore \tfrac{1}{2} c &= 21^\circ\ 33'\ 37'' & \therefore \tfrac{1}{2} C &= 25^\circ\ 48'\ 58'' \\ \therefore c &= 43^\circ\ \ 7'\ 14'' & \therefore C &= 51^\circ\ 37'\ 56'' \end{aligned}$$

Exercise 102. Given Two Sides and an Opposite Angle

1. Given $a = 75^\circ$, $b = 110^\circ$, $A = 85^\circ$, find B.
2. Given $b = 80^\circ$, $c = 115^\circ$, $B = 95^\circ$, find C.
3. Given $c = 95^\circ$, $a = 120^\circ$, $C = 97^\circ$, find A.

Solve the triangles, given the following parts:

4. $a = 73^\circ\ 49'\ 38''$, $b = 120^\circ\ 53'\ 35''$, $A = 88^\circ\ 52'\ 42''$.
5. $a = 150^\circ\ 57'\ 5''$, $b = 134^\circ\ 15'\ 54''$, $A = 144^\circ\ 22'\ 42''$.
6. $a = 79^\circ\ 0'\ 54''$, $b = 82^\circ\ 17'\ 4''$, $A = 82^\circ\ 9'\ 26''$.
7. Given $a = 30^\circ\ 52'\ 37''$, $b = 31^\circ\ 9'\ 16''$, and $A = 87^\circ\ 34'\ 12''$, show that the triangle is impossible.

Reviewing preceding work, find the value of the third angle, given:

8. $A = 130^\circ\ 17'$, $B = 78^\circ\ 19'$, $c = 48^\circ\ 32'$.
9. $B = 142^\circ\ 20'$, $C = 79^\circ\ 56'$, $a = 82^\circ\ 18'$.
10. $B = 156^\circ\ 15'$, $C = 83^\circ\ 26'$, $a = 75^\circ\ 48'$.
11. $C = 75^\circ\ 48'$, $A = 132^\circ\ 17'$, $b = 64^\circ\ 19'$.
12. $C = 83^\circ\ 52'$, $A = 127^\circ\ 48'$, $b = 72^\circ\ 50'$.
13. $A = 36.75^\circ$, $B = 48.25^\circ$, $c = 132.5^\circ$.
14. $A = 48.5^\circ$, $B = 62.125^\circ$, $c = 128.75^\circ$.
15. $B = 156.6^\circ$, $b = 95.7^\circ$, $c = 117.8^\circ$.

Reviewing preceding work, solve the following triangles:

16. $B = 153^\circ\ 17'\ 6''$, $C = 78^\circ\ 43'\ 36''$, $a = 86^\circ\ 15'\ 15''$.
17. $A = 125^\circ\ 41'\ 44''$, $C = 82^\circ\ 47'\ 35''$, $b = 52^\circ\ 37'\ 57''$.

202. Given Two Angles and a Side opposite one of them. For example, given A, B, and a, solve the triangle.

From the Law of Sines (§ 189),

$$\sin b = \frac{\sin a \sin B}{\sin A},$$

whence b can be found, a, B, and A being given.

We may now find c and C from the formulas of § 195, written thus:

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b),$$

and

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B).$$

In this case the conditions for one solution, two solutions, or no solution can be deduced directly by the theory of polar triangles from the corresponding conditions of § 201. There are *two solutions* when A and a are alike in kind and $\sin B > \sin A > \sin a \sin B$; *no solution* when A and a are unlike in kind (including the case in which either A or a is 90°) and $\sin B > \sin A$ or $\sin B = \sin A$, or when $\sin A < \sin a \sin B$; and *one solution* in every other case.

By proceeding as indicated in § 201, formulas for computing c or C, independent of the side b, may be found; namely,

$$\tan m = \tan a \cos B, \text{ and } \sin(c-m) = \cot A \tan B \sin m;$$
$$\cot x = \cos a \tan B, \text{ and } \sin(C-x) = \cos A \sec B \sin x.$$

In these formulas $m = BD$, $x = \angle BCD$, D being the foot of the perpendicular from the vertex C.

Only those values of b can be retained which are greater than or less than a, according as B is greater than or less than A. If $\log \sin b$ is positive, the triangle is impossible.

Exercise 103. Given Two Angles and an Opposite Side

Solve the triangles, given the following parts:

1. $A = 110°, B = 130°, a = 150°$.
2. $A = 120°, B = 115°, a = 70°$.
3. $A = 100°, B = 100°, a = 90°$.
4. $A = 95°, B = 96°, a = 100°$.
5. $B = 98°, C = 105°, b = 80°$.
6. $C = 92°, A = 115°, c = 95°$.

Find the side b, given the following parts:

7. $A = 110° 10'$, $B = 133° 18'$, $a = 147° 5' 32''$.
8. $B = 113° 39' 21''$, $C = 123° 40' 18''$, $b = 65° 39' 46''$.
9. $C = 100° 2' 11''$, $A = 98° 30' 28''$, $c = 95° 20' 39''$.
10. $B = 105° 13' 42''$, $C = 110° 37' 35''$, $b = 78°, 75' 12''$.

11. Given $A = 24° 33' 9''$, $B = 38° 0' 12''$, and $a = 65° 20' 13''$, show that the triangle is impossible.

203. Given the Three Sides. In this case we have given a, b, and c, to solve the triangle. From § 192 we have the formula

$$\tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}},$$

where $s=\frac{1}{2}(a+b+c)$. Hence A can be found, a, b, and c being given. The results may then be checked by the Law of Sines (§ 189).

The formulas for $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ may be used, but in general the one for $\tan\frac{1}{2}A$ is more satisfactory, because the tangent varies more rapidly.

For example, given $a=124°\,12'\,31''$, $b=54°\,18'\,16''$, $c=97°\,12'\,25''$, solve the triangle.

$$\begin{aligned}
a &= 124°\,12'\,31''\\
b &= 54°\,18'\,16''\\
c &= \underline{97°\,12'\,25''}\\
2s &= 275°\,43'\,12''\\
\therefore s &= 137°\,51'\,36''
\end{aligned}
\qquad
\begin{aligned}
s-a &= 13°\,39'\,5''\\
s-b &= 83°\,33'\,20''\\
s-c &= 40°\,39'\,11''
\end{aligned}$$

$$\begin{aligned}
\log\sin(s-b) &= 9.99725\\
\log\sin(s-c) &= 9.81390\\
\operatorname{colog}\sin s &= 0.17331\\
\operatorname{colog}\sin(s-a) &= \underline{0.62707}\\
2\,)\,&\underline{0.61153}\\
\log\tan\tfrac{1}{2}A &= 0.30577
\end{aligned}
\qquad
\begin{aligned}
\log\tan\tfrac{1}{2}A &= 0.30577\\
\log\tan\tfrac{1}{2}B &= 9.68145\\
\log\tan\tfrac{1}{2}C &= 9.86480\\
\therefore \tfrac{1}{2}A &= 63°\,41'\,3.8''\\
\therefore \tfrac{1}{2}B &= 25°\,39'\,5.6''\\
\therefore \tfrac{1}{2}C &= \underline{36°\,13'\,20''}\\
\therefore A &= 127°\,22'\,8''\\
\therefore B &= 51°\,18'\,11''\\
\therefore C &= 72°\,26'\,40''
\end{aligned}$$

Similarly for B and C.

Check.

$$\begin{aligned}
\log\sin a &= 9.91750\\
\log\sin A &= \underline{9.90023}\\
&\ 0.01727
\end{aligned}
\qquad
\begin{aligned}
\log\sin b &= 9.90962\\
\log\sin B &= \underline{9.89235}\\
&\ 0.01727
\end{aligned}
\qquad
\begin{aligned}
\log\sin c &= 9.99656\\
\log\sin C &= \underline{9.97929}\\
&\ 0.01727
\end{aligned}$$

Exercise 104. Given the Three Sides

Solve the triangles, given the following parts:

1. $a=120°$, $b=60°$, $c=110°$.
2. $a=50°$, $b=115°$, $c=130°$.
3. $a=130°$, $b=110°$, $c=85°$.
4. $a=20°$, $b=60°$, $c=70°$.
5. $a=30°$, $b=50°$, $c=80°$.
6. $a=55°$, $b=100°$, $c=125°$.

Find the value of A, given the following parts:

7. $a=120°\,55'\,35''$, $b=59°\,4'\,25''$, $c=106°\,10'\,22''$.
8. $a=50°\,12'\,4''$, $b=116°\,44'\,48''$, $c=129°\,11'\,42''$.
9. $a=131°\,35'\,4''$, $b=108°\,30'\,14''$, $c=84°\,46'\,34''$.
10. $a=20°\,16'\,38''$, $b=56°\,19'\,40''$, $c=66°\,20'\,44''$.

204. Given the Three Angles. In this case we have given the three angles, A, B, and C, to solve the triangle.

From § 193 we have the formula

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}},$$

where $S = \frac{1}{2}(A+B+C)$. Hence a can be found, A, B, and C being given. The results may then be checked by the Law of Sines (§ 189).

As in § 203, the formula for $\tan \frac{1}{2} a$ is to be preferred to those for $\sin \frac{1}{2} a$ or $\cos \frac{1}{2} a$, because the tangent varies more rapidly than the sine or cosine.

For example, given $A = 220°$, $B = 130°$, $C = 150°$, find a.

$$\begin{array}{rlrl}
A = & 220° & \log \cos S = & 9.53405\ (n) \\
B = & 130° & \log \cos (S-A) = & 9.93753 \\
C = & \underline{150°} & \text{colog} \cos (S-B) = & 0.30103\ (n) \\
2S = & 500° & \text{colog} \cos (S-C) = & \underline{0.76033\ (n)} \\
\therefore S = & 250° & 2\,) & \underline{0.53294} \\
S-A = & 30° & \log \tan \frac{1}{2} a = & 0.26647 \\
S-B = & 120° & \therefore \frac{1}{2} a = & 61°\ 34'\ 6'' \\
S-C = & 100° & \therefore a = & 123°\ 8'\ 12''
\end{array}$$

Here (n) indicates that the factor is negative, $\cos S$ being $\cos 250°$ and therefore negative. The three negative factors, with the negative sign before the product, make the result positive.

In the same way we may find b and c, checking the work by the Law of Sines, as in § 203.

Exercise 105. Given the Three Angles

Solve the triangles, given the following parts:

1. $A = 120°$, $B = 112°$, $C = 85°$.
2. $A = 60°$, $B = 80°$, $C = 60°$.
3. $A = 100°$, $B = 55°$, $C = 92°$.
4. $A = 5°$, $B = 39°$, $C = 150°$.
5. $A = 75°$, $B = 75°$, $C = 75°$.
6. $A = 100°$, $B = 105°$, $C = 110°$.

Find a and b, given the following parts:

7. $A = 130°$, $B = 110°$, $C = 80°$.
8. $A = 59°\ 55'\ 10''$, $B = 85°\ 36'\ 50''$, $C = 59°\ 55'\ 10''$.
9. $A = 102°\ 14'\ 12''$, $B = 54°\ 32'\ 24''$, $C = 89°\ 5'\ 46''$.
10. $A = 4°\ 23'\ 35''$, $B = 8°\ 28'\ 20''$, $C = 172°\ 17'\ 56''$.
11. $A = 71°\ 27'\ 30''$, $B = 16°\ 29'\ 30''$, $C = 140°\ 18'\ 50''$.
12. $A = 42.75°$, $B = 27.5°$, $C = 150.3°$.
13. $A = 72.51°$, $B = 142.65°$, $C = 100.2°$.
14. $A = 121°\ 10'\ 10''$, $B = 68°\ 42'\ 30''$, $C = 21°\ 17'\ 30''$.

205. Area of a Spherical Triangle. A spherical triangle is equivalent to a lune whose angle is half the spherical excess of the triangle.

See the Wentworth-Smith Plane and Solid Geometry, § 695. If the angles are A, B, and C, the spherical excess (E) is $A + B + C - 180°$.

For example, to find the area of a triangle whose angles are 110°, 100°, and 95°, on the surface of a sphere whose radius is 6 in.

$$\text{Spherical excess} = 110° + 100° + 95° - 180° = 125°.$$

Hence $\quad$ angle of lune $= 62\frac{1}{2}°$.

Therefore $\quad$ area of lune $= \dfrac{62\frac{1}{2}}{360}$ of the spherical surface

$$= \frac{62\frac{1}{2}}{360} \times 4 \times 3.1416 \times 36 \text{ sq. in.}$$

Therefore area of triangle $= 78.54$ sq. in.

That is, the area (T) of the triangle equals $\dfrac{\frac{1}{2}E}{360} \cdot 4\pi r^2$.

$$\therefore\ T = \frac{E\pi r^2}{180}.$$

In case the three angles are not given, they may be found by solving the triangle from the parts that are known. In case the three sides are given, however, it is possible to find E directly by means of Lhuilier's Formula (§ 206).

For example, given $A = 102° 14' 12''$, $B = 54° 32' 24''$, $C = 89° 5' 46''$.

$$\begin{array}{rl}
A = & 102° 14' 12'' \\
B = & 54° 32' 24'' \\
C = & \underline{89°\ \ 5' 46''} \\
 & 245° 52' 22'' \\
\therefore E = & 65° 52' 22'' \\
= & 237{,}142'' \\
180° = & 648{,}000''
\end{array}
\qquad
\begin{array}{rl}
\log r^2 = & \log r^2 \\
\log E = & 5.37501 \\
\log \pi = & 0.49715 \\
\text{colog } 648{,}000 = & \underline{4.18842 - 10} \\
\log T = & 0.06058 + \log r^2 \\
\therefore T = & 1.1497\, r^2
\end{array}$$

Hence, if we know the radius of the sphere, we can express the area of a spherical triangle in the ordinary units of area.

Exercise 106. Areas of Spherical Triangles

Find the areas of the following triangles:

1. $A = 80°$, $B = 35°$, $C = 70°$, $r = 10$.
2. $A = 85° 30'$, $B = 29° 45'$, $C = 72° 15'$, $r = 5$.
3. $A = 84° 20' 19''$, $B = 27° 22' 40''$, $C = 75° 33'$, $r = 20$.
4. $A = 93° 30' 10''$, $B = 32° 35' 30''$, $C = 88° 25'$, $r = 50$.

206. Lhuilier's Formula. In case the three sides of a spherical triangle are given, it is possible to find the spherical excess directly by means of the following ingenious formula given by the Swiss mathematician, Lhuilier (1750–1840),

$$\tan^2 \tfrac{1}{4}E = \tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c).$$

The formula is deduced as follows:

From § 194, $$\frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}C} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c},$$

and, from § 8, $$\sin \tfrac{1}{2}C = \cos(90° - \tfrac{1}{2}C).$$

Therefore $$\frac{\cos \frac{1}{2}(A+B)}{\cos(90° - \frac{1}{2}C)} = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c}.$$

Then, by division and composition,

$$\frac{\cos \frac{1}{2}(A+B) - \cos(90° - \frac{1}{2}C)}{\cos \frac{1}{2}(A+B) + \cos(90° - \frac{1}{2}C)} = \frac{\cos \frac{1}{2}(a+b) - \cos \frac{1}{2}c}{\cos \frac{1}{2}(a+b) + \cos \frac{1}{2}c}. \tag{1}$$

Furthermore, by dividing in § 103, we see that

$$\frac{\cos A - \cos B}{\cos A + \cos B} = -\tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B). \tag{2}$$

Substituting in (2) for A and B the values $\frac{1}{2}(A+B)$ and $90° - \frac{1}{2}C$ respectively, we have

$$\begin{aligned}
&\frac{\cos \frac{1}{2}(A+B) - \cos(90° - \frac{1}{2}C)}{\cos \frac{1}{2}(A+B) + \cos(90° - \frac{1}{2}C)} \\
&\quad = -\tan \tfrac{1}{2}(\tfrac{1}{2}A + \tfrac{1}{2}B + 90° - \tfrac{1}{2}C) \tan \tfrac{1}{2}(\tfrac{1}{2}A + \tfrac{1}{2}B - 90° + \tfrac{1}{2}C) \\
&\quad = -\tan \tfrac{1}{4}(A+B-C+180°) \tan \tfrac{1}{4}(A+B+C-180°).
\end{aligned}$$

We see that the angle in the last factor in this formula is the spherical excess of the triangle, and we now introduce the symbol for this excess; namely,

$$E = A + B + C - 180°.$$

$$\begin{aligned}
\therefore \tan \tfrac{1}{4}(A+B-C+180°) &= \tan \tfrac{1}{4}(360° - 2C + A + B + C - 180°) \\
&= \tan \tfrac{1}{4}(360° - 2C + E) \\
&= \tan [90° - \tfrac{1}{4}(2C - E)] \\
&= \cot \tfrac{1}{4}(2C - E).
\end{aligned}$$

Substituting E for $A + B + C - 180°$ and $\cot \frac{1}{4}(2C - E)$ for $\tan \frac{1}{4}(A + B - C + 180°)$, we have

$$\frac{\cos \frac{1}{2}(A+B) - \cos(90° - \frac{1}{2}C)}{\cos \frac{1}{2}(A+B) + \cos(90° - \frac{1}{2}C)} = -\cot \tfrac{1}{4}(2C - E) \tan \tfrac{1}{4}E. \tag{3}$$

Substituting in (2) for A and B the values $\frac{1}{2}(a+b)$ and $\frac{1}{2}c$, and also substituting s for $\frac{1}{2}(a+b+c)$ and $s-c$ for $\frac{1}{2}(a+b-c)$, we have

$$\frac{\cos\frac{1}{2}(a+b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}=-\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c). \qquad (4)$$

Comparing (1), (3), and (4) we obtain

$$\cot\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-c). \qquad (5)$$

By beginning with the second of Gauss's equations (§ 194), and treating it in the same way, we obtain as the result

$$\tan\tfrac{1}{4}(2C-E)\tan\tfrac{1}{4}E=\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b) \qquad (6)$$

By taking the product of (5) and (6) we obtain the formula given on page 224 and known as Lhuilier's Formula.

By means of this formula, E can be computed from the three sides much more easily than by first finding the angles, and then the area of the triangle can be found by § 205.

For example, given $a=133°\,26'\,19''$, $b=64°\,50'\,53''$, $c=144°\,13'\,45''$, find E.

$$\begin{aligned}
a &= 133°\,26'\,19'' & \log\tan\tfrac{1}{2}s &= 1.11669\\
b &= 64°\,50'\,53'' & \log\tan\tfrac{1}{2}(s-a) &= 9.53474\\
c &= \underline{144°\,13'\,45''} & \log\tan\tfrac{1}{2}(s-b) &= 0.12612\\
2s &= \underline{342°\,30'\,57''} & \log\tan\tfrac{1}{2}(s-c) &= \underline{9.38083}\\
s &= 171°\,15'\,28.5'' & \log\tan^2\tfrac{1}{4}E &= 0.15838\\
s-a &= 37°\,49'\,9.5'' & \log\tan\tfrac{1}{4}E &= 0.07919\\
s-b &= 106°\,24'\,35.5'' & \therefore \tfrac{1}{4}E &= 50°\,11'\,41.5''\\
s-c &= 27°\,1'\,43.5'' & \therefore E &= 200°\,46'\,46''
\end{aligned}$$

Exercise 107. Finding Areas

Find the spherical excess, given:

1. $A=80°$, $B=30°$, $C=75°$.
2. $A=70°$, $B=110°$, $C=80°$.
3. $A=95°$, $B=120°$, $C=85°$.
4. $A=88°$, $B=95°$, $C=100°$.
5. $A=72°$, $B=98°$, $C=110°$.
6. $A=96°$, $B=97°$, $C=98°$.

Find the areas of the following triangles, given:

7. $a=100°$, $b=75°$, $c=80°$.
8. $a=110°$, $b=85°$, $c=95°$.
9. $A=120°$, $B=78°$, $c=115°$.
10. $A=60°$, $a=75°$, $b=80°$.
11. $A=80°$, $B=75°$, $a=75°$.
12. $A=150°$, $b=45°$, $c=15°$.
13. $A=85°$, $C=95°$, $b=70°$.
14. $B=75°$, $b=72°$, $c=55°$.

Exercise 108. Miscellaneous Examples

Find the spherical excess, given:

1. $A = 84° 20' 19''$, $B = 27° 22' 40''$, $C = 75° 33'$.
2. $a = 69° 15' 6''$, $b = 120° 42' 47''$, $c = 159° 18' 33''$.
3. $a = 33° 1' 45''$, $b = 155° 5' 18''$, $C = 110° 10'$.

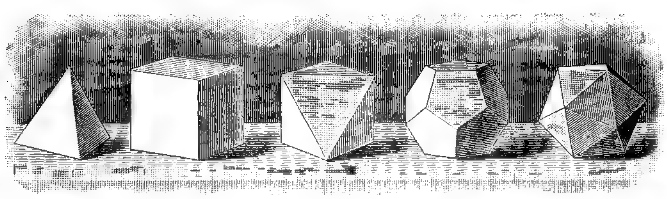
Find the areas of the following triangles, given:

4. $c = 114° 27' 57''$, $A = 78° 42' 33''$, $B = 127° 13' 7''$.
5. $a = 76° 14' 47''$, $b = 82° 40' 15''$, $A = 60° 22' 44''$.
6. $A = 80° 12' 35''$, $B = 77° 38' 22''$, $a = 76° 42' 28''$.
7. $b = 44° 27' 40''$, $c = 15° 22' 44''$, $A = 167° 42' 27''$.
8. $b = 67° 15' 42''$, $A = 84° 55' 8''$, $C = 96° 18' 49''$.
9. $b = 72° 19' 38''$, $c = 54° 58' 52''$, $B = 77° 15' 14''$.
10. $B = 127° 16' 4''$, $C = 42° 34' 19''$, $b = 54° 47' 55''$.
11. $a = 128° 42' 56''$, $b = 107° 13' 48''$, $c = 88° 37' 51''$.
12. $A = 127° 22' 28''$, $B = 131° 45' 27''$, $C = 100° 52' 16''$.
13. $a = 116° 19' 45''$, $A = 160° 42' 24''$, $C = 171° 27' 15''$.

14. Find the area of a triangle on the surface of the earth, regarded as a sphere, if each side of the triangle is equal to 1°, and the radius of the earth is taken as 3958 mi.

15. In an equilateral triangle, given the side a, find the angle A.

16. Given the side a of a regular spherical polygon of n sides, find the angle A of the polygon, the distance R from the center of the polygon to one of the vertices, and the distance r from the center to the middle point of one of the sides.



17. Compute the dihedral angles made by the faces of the five regular polyhedrons.

18. The distance from Washington (W) to a certain place X, measured in degrees on a great-circle arc, is 9°, and of a place Y from Washington the distance is 12°. The angle XWY is 85°. What is the distance in degrees from X to Y?

THE MOST IMPORTANT FORMULAS OF SPHERICAL TRIGONOMETRY

Principal Formulas of Right Triangles (§§ 172–174)

$\cos c = \cos a \cos b.$	$\cos A = \cos a \sin B.$
$\sin a = \sin c \sin A.$	$\cos B = \cos b \sin A.$
$\sin b = \sin c \sin B.$	$\sin b = \tan a \cot A.$
$\cos A = \tan b \cot c.$	$\sin a = \tan b \cot B.$
$\cos B = \tan a \cot c.$	$\cos c = \cot A \cot B.$

Auxiliary Formulas of Right Triangles (§ 175)

$$\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2}(c - a) \tan \tfrac{1}{2}(c + a).$$

$$\tan^2 (45° - \tfrac{1}{2} A) = \tan \tfrac{1}{2}(c - a) \cot \tfrac{1}{2}(c + a).$$

$$\tan^2 \tfrac{1}{2} B = \frac{\sin (c - a)}{\sin (c + a)}.$$

$$\tan^2 \tfrac{1}{2} c = \frac{-\cos (A + B)}{\cos (A - B)}.$$

$$\tan^2 \tfrac{1}{2} a = \tan [\tfrac{1}{2}(A + B) - 45°] \tan [\tfrac{1}{2}(A - B) + 45°].$$

$$\tan^2 (45° - \tfrac{1}{2} c) = \tan \tfrac{1}{2}(A - a) \cot \tfrac{1}{2}(A + a).$$

$$\tan^2 (45° - \tfrac{1}{2} b) = \frac{\sin (A - a)}{\sin (A + a)}.$$

$$\tan^2 (45° - \tfrac{1}{2} B) = \tan \tfrac{1}{2}(A - a) \tan \tfrac{1}{2}(A + a).$$

Napier's Rules (§ 176)

1. *The sine of any middle part is equal to the product of the tangents of the adjacent parts.*
2. *The sine of any middle part is equal to the product of the cosines of the opposite parts.*

Principal Formulas of Oblique Triangles (§§ 189–191)

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B.$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

$$\cos A = - \cos B \cos C + \sin B \sin C \cos a.$$

$$\cos B = - \cos A \cos C + \sin A \sin C \cos b.$$

$$\cos C = - \cos A \cos B + \sin A \sin B \cos c.$$

Auxiliary Formulas of Oblique Triangles (§§ 192, 193)

$$\sin\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

$$\cos\tfrac{1}{2}A=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}.$$

$$\tan\tfrac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

And similarly for the sine, cosine, and tangent of B and C.

$$\sin\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}}.$$

$$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}.$$

$$\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

And similarly for the sine, cosine, and tangent of b and c.

Gauss's Equations (§ 194)

$$\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$
$$\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.$$
$$\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$
$$\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.$$

Napier's Analogies (§ 195)

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\tfrac{1}{2}(A-B)}{\cos\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\tfrac{1}{2}(A-B)}{\sin\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c.$$

Areas of Triangles (§ 205)

$$T=\frac{E\pi r^2}{180},\quad\text{where}\quad E=A+B+C-180^\circ.$$

Lhuilier's Formula (§ 206)

$$\tan^2\tfrac{1}{4}E=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c).$$

INDEX

ANSWERS

ANSWERS

PLANE TRIGONOMETRY

Exercise 1. Page 5

1. $\cos B = \frac{a}{c}$; $\tan B = \frac{b}{a}$; $\cot B = \frac{a}{b}$; $\sec B = \frac{c}{a}$; $\csc B = \frac{c}{b}$.

3. $\tan A$. 4. $\cot A$. 5. $\sec A$. 6. $\csc A$.

7. $\sin A = \frac{3}{5}$; $\cos A = \frac{4}{5}$; $\tan A = \frac{3}{4}$; $\cot A = \frac{4}{3}$; $\sec A = \frac{5}{4}$; $\csc A = \frac{5}{3}$.

8. $\sin A = \frac{5}{13}$; $\cos A = \frac{12}{13}$; $\tan A = \frac{5}{12}$; $\cot A = \frac{12}{5}$; $\sec A = \frac{13}{12}$; $\csc A = \frac{13}{5}$.

9. $\sin A = \frac{8}{17}$; $\cos A = \frac{15}{17}$; $\tan A = \frac{8}{15}$; $\cot A = \frac{15}{8}$; $\sec A = \frac{17}{15}$; $\csc A = \frac{17}{8}$.

10. $\sin A = \frac{9}{41}$; $\cos A = \frac{40}{41}$; $\tan A = \frac{9}{40}$; $\cot A = \frac{40}{9}$; $\sec A = \frac{41}{40}$; $\csc A = \frac{41}{9}$.

11. $\sin A = \frac{39}{89}$; $\cos A = \frac{80}{89}$; $\tan A = \frac{39}{80}$; $\cot A = \frac{80}{39}$; $\sec A = \frac{89}{80}$; $\csc A = \frac{89}{39}$.

12. $\sin A = \frac{119}{169}$; $\cos A = \frac{120}{169}$; $\tan A = \frac{119}{120}$; $\cot A = \frac{120}{119}$; $\sec A = \frac{169}{120}$; $\csc A = \frac{169}{119}$.

13. $a^2 + b^2 = c^2$.

14. $\sin A = \dfrac{2n}{n^2+1}$; $\cos A = \dfrac{n^2-1}{n^2+1}$; $\tan A = \dfrac{2n}{n^2-1}$; $\cot A = \dfrac{n^2-1}{2n}$; $\sec A = \dfrac{n^2+1}{n^2-1}$; $\csc A = \dfrac{n^2+1}{2n}$.

15. $\sin A = \dfrac{2n}{n^2+1}$; $\cos A = \dfrac{n^2-1}{n^2+1}$; $\tan A = \dfrac{2n}{n^2-1}$; $\cot A = \dfrac{n^2-1}{2n}$; $\sec A = \dfrac{n^2+1}{n^2-1}$; $\csc A = \dfrac{n^2+1}{2n}$.

16. $\sin A = \dfrac{2mn}{m^2+n^2}$; $\cos A = \dfrac{m^2-n^2}{m^2+n^2}$; $\tan A = \dfrac{2mn}{m^2-n^2}$; $\cot A = \dfrac{m^2-n^2}{2mn}$; $\sec A = \dfrac{m^2+n^2}{m^2-n^2}$; $\csc A = \dfrac{m^2+n^2}{2mn}$.

17. $\sin A = \dfrac{2mn}{m^2+n^2}$; $\cos A = \dfrac{m^2-n^2}{m^2+n^2}$; $\tan A = \dfrac{2mn}{m^2-n^2}$; $\cot A = \dfrac{m^2-n^2}{2mn}$; $\sec A = \dfrac{m^2+n^2}{m^2-n^2}$; $\csc A = \dfrac{m^2+n^2}{2mn}$.

19. $\sin A = \frac{1}{2}\sqrt{2} = \cos A$; $\tan A = 1 = \cot A$; $\sec A = \sqrt{2} = \csc A$.

20. $\sin A = \frac{2}{5}\sqrt{5}$; $\cos A = \frac{1}{5}\sqrt{5}$; $\tan A = 2$; $\cot A = \frac{1}{2}$; $\sec A = \sqrt{5}$; $\csc A = \frac{1}{2}\sqrt{5}$.

21. $\sin A = \frac{2}{3}$; $\cos A = \frac{1}{3}\sqrt{5}$; $\tan A = \frac{2}{5}\sqrt{5}$; $\cot A = \frac{1}{2}\sqrt{5}$; $\sec A = \frac{3}{5}\sqrt{5}$; $\csc A = \frac{3}{2}$.

22. $\sin B = \frac{143}{145}$; $\cos B = \frac{24}{145}$; $\tan B = \frac{143}{24}$; $\cot B = \frac{24}{143}$; $\sec B = \frac{145}{24}$; $\csc B = \frac{145}{143}$.

23. $\sin B = \frac{95}{193}$; $\cos B = \frac{168}{193}$; $\tan B = \frac{95}{168}$; $\cot B = \frac{168}{95}$; $\sec B = \frac{193}{168}$; $\csc B = \frac{193}{95}$.

24. $\sin B = \frac{23}{265}$; $\cos B = \frac{264}{265}$; $\tan B = \frac{23}{264}$; $\cot B = \frac{264}{23}$; $\sec B = \frac{265}{264}$; $\csc B = \frac{265}{23}$.

25. $\sin B = \dfrac{2\sqrt{pq}}{p+q}$; $\cos B = \dfrac{p-q}{p+q}$; $\tan B = \dfrac{2\sqrt{pq}}{p-q}$; $\cot B = \dfrac{p-q}{2pq}\sqrt{pq}$;
$\sec B = \dfrac{p+q}{p-q}$; $\csc B = \dfrac{p+q}{2pq}\sqrt{pq}$.

26. $\sin A = \dfrac{\sqrt{p^2+q^2}}{p+q} = \cos B$; $\cot A = \dfrac{\sqrt{2pq}}{\sqrt{p^2+q^2}} = \tan B$;
$\cos A = \dfrac{\sqrt{2pq}}{p+q} = \sin B$; $\sec A = \dfrac{p+q}{\sqrt{2pq}} = \csc B$;
$\tan A = \dfrac{\sqrt{p^2+q^2}}{\sqrt{2pq}} = \cot B$; $\csc A = \dfrac{p+q}{\sqrt{p^2+q^2}} = \sec B$.

27. $\sin A = \dfrac{\sqrt{p^2+p}}{p+1} = \cos B$; $\cot A = \dfrac{\sqrt{p}}{p} = \tan B$;
$\cos A = \dfrac{1}{\sqrt{p+1}} = \sin B$; $\sec A = \sqrt{p+1} = \csc B$;
$\tan A = \sqrt{p} = \cot B$; $\csc A = \dfrac{\sqrt{p^2+p}}{p} = \sec B$.

28. 12.3.
29. 1.54.
30. 9.
31. 6800.
32. 4000.
33. 227.84.
34. $3\sqrt{13}$; 9.
35. $\frac{7}{3}\sqrt{3}$; $\frac{7}{6}\sqrt{3}$.
36. 5; 3.
37. 2.5; 1.5.
38. 1.5 mi.; 2 mi.
40. $a = 0.342$; $b = 0.94$.
41. $a = 1.368$; $b = 3.76$.
42. $a = 1.197$; $b = 3.29$.
43. $a = 1.6416$; $b = 4.512$.
44. $a = 2.565$; $b = 7.05$.
45. $a = 0.643$; $b = 0.766$.
46. $a = 1.929$; $b = 2.298$.
47. $a = 4.501$; $b = 5.362$.
48. $a = 6.8801$; $b = 8.1962$
49. $a = 160.75$; $b = 191.5$.
50. $a = 1.88$; $b = 0.684$.
51. $c = 2.128$; $b = 0.728$.
52. $c = 5.848$; $a = 5.494$.
53. $c = 26.6$; $b = 9.1$.
54. $a = 412.05$; $c = 438.6$.
55. 142.926 yd.
56. $1\frac{1}{7}$; 24 ft.

Exercise 2. Page 7

1. cos 60°. **2.** sin 70°. **3.** cot 50°. **4.** csc 65°. **5.** cos 40°. **6.** cot 30°. **7.** csc 15°. **8.** sec 5°. **9.** cos 30°. **10.** sin 30°. **11.** cot 45°. **12.** csc 45°. **13.** cos 14° 30′. **14.** cot 7° 15′. **15.** csc 21° 45′. **16.** sin 1° 50′. **17.** cos 25°. **18.** cot 10°. **19.** csc 13°. **20.** sin 38°. **21.** tan 29°. **22.** sec 12°. **23.** cos 1°. **24.** sin 4°. **25.** csc 2°. **26.** cos $12\frac{1}{2}$°. **27.** sin $7\frac{1}{2}$°. **28.** cot 1.4°. **29.** 45°. **30.** 45°. **31.** 30°. **32.** 30°.

Exercise 3. Page 9

1. 0.5. **2.** 0.8660. **3.** 0.5773. **4.** 1.7320. **5.** 1.1547. **6.** 2. **7.** 0.8660. **8.** 0.5. **9.** 1.7320. **10.** 0.5773. **11.** 2. **12.** 1.1547. **13.** $\sqrt{2}$. **14.** $\frac{1}{3}\sqrt{6}$. **15.** $\sqrt{3}$. **16.** $\frac{1}{3}\sqrt{3}$. **17.** $\frac{1}{2}\sqrt{6}$. **18.** $\frac{1}{2}\sqrt{2}$. **19.** $\frac{1}{3}\sqrt{3}$. **20.** $\sqrt{3}$. **21.** $\frac{1}{3}$. **22.** 3. **23.** $\frac{1}{3}\sqrt{3}$. **24.** $\sqrt{3}$.

25. cos 27° 42′ 20″. **26.** cot 14° 31′ 25″. **27.** csc 2° 27′ 9″. **28.** sin 1° 59′ 33″. **29.** cos 14.2°. **30.** sin 7.25°. **31.** cot 21.18°. **32.** csc 4.05°.

33. 90°. **34.** 60°. **35.** 22° 30′. **36.** 18°. **37.** $\dfrac{90°}{n+1}$. **38.** 90°. **39.** 60°. **40.** 22° 30′. **41.** 18°. **42.** $\dfrac{90°}{n+1}$. **43.** $\frac{1}{3}\sqrt{6}$. **44.** $\sqrt{2}$. **45.** $\sqrt{6}$. **46.** $\frac{2}{3}\sqrt{3}$. **47.** $2\sqrt{3}$. **48.** 2. **49.** $\frac{1}{3}\sqrt{3}$. **50.** $\frac{1}{3}\sqrt{3}$. **51.** 1. **52.** $\frac{1}{3}\sqrt{3}$.

Exercise 4. Page 10

1. 0.0872.	**7.** 0.3584.	**13.** 0.9135.	**19.** 5.1446.	**25.** 1.0000.	**31.** 1.4396.
2. 0.2419.	**8.** 0.5000.	**14.** 0.9135.	**20.** 5.1446.	**26.** 1.0000.	**32.** 1.4396.
3. 0.3584.	**9.** 0.9945.	**15.** 0.8192.	**21.** 0.3839.	**27.** 1.0353.	**33.** 0.0038.
4. 0.5000.	**10.** 0.9945.	**16.** 0.8192.	**22.** 0.3839.	**28.** 1.0353.	**34.** 0.0054.
5. 0.0872.	**11.** 0.9703.	**17.** 11.4301.	**23.** 1.0000.	**29.** 4.8097.	**35.** 2 sec 10°.
6. 0.2419.	**12.** 0.9703.	**18.** 11.4301.	**24.** 1.0000.	**30.** 4.8097.	**36.** 2 csc 10°.

37. 2 cos 15°.

38. $3 \sin 20° > \sin(3 \times 20°)$ and $> \sin(2 \times 20°)$.

39. $3 \tan 10° < \tan(3 \times 10°)$ and $> \tan(2 \times 10°)$.

40. $3 \cos 10° > \cos(3 \times 10°)$ and $> \cos(2 \times 10°)$.

41. No.

42. The sin, tan, sec increase and the cos, cot, csc decrease.

Exercise 5. Page 12

12. 37.6.	**13.** 1.	**14.** 100.	**15.** 60.	**16.** 12.86.	**17.** 22.64.

Exercise 6. Page 15

1. 1.736.	**4.** 57.45.	**7.** 39°.	**10.** 54 ft.	**13.** 449.9 ft.
2. 3.882.	**5.** 12°.	**8.** 43°.	**11.** 4.326 ft.	
3. 41.01.	**6.** 20°.	**9.** 30°.	**12.** 479.9 ft.	

Exercise 7. Page 16

1. 10.83.	**8.** 5.935.	**15.** 63°.	**22.** 411.4 ft.	**29.** 6 in.
2. 13.46.	**9.** 4.884.	**16.** 70°.	**23.** 383 ft.	**30.** 28.19 ft.; 21.21 ft.;
3. 25.58.	**10.** 7.311.	**17.** 54°.	**24.** 43°.	12.68 ft.; 30 ft.; 0 ft.
4. 31.86.	**11.** 10°.	**18.** 60°.	**25.** 7.794 in.	**31.** 60°; 0°.
5. 55.73.	**12.** 17°.	**19.** 70°.	**26.** 166.272 sq. in.	**32.** 25°; 65°.
6. 1.873.	**13.** 26°.	**20.** 84°.	**27.** 5.657.	**33.** 30° and 60°;
7. 5.972.	**14.** 60°.	**21.** 60°.	**28.** 27.71 ft.	31° and 59°.
				34. 749.9 ft.

Exercise 8. Page 19

1. 12.02.	**6.** 5.928.	**11.** 45°.	**16.** 64°.	**20.** 159.7 ft.
2. 11.04.	**7.** 14.78.	**12.** 8°.	**17.** 148 ft. 8 in.	**21.** 45°; 90°; 45°.
3. 28.84.	**8.** 44.01.	**13.** 9°.	**18.** 29°.	**22.** 15.76 ft.
4. 45.04.	**9.** 107.1.	**14.** 19°.	**19.** 2.517 mi;	**23.** 6.14 ft.
5. 98.	**10.** 453.8.	**15.** 22°.	3.916 mi.	**24.** 1.03 in.

Exercise 9. Page 20

1. 26.11.	**4.** 85.81.	**7.** 26.60.	**10.** 25°.	**13.** 113 ft.
2. 12.35.	**5.** 544.0.	**8.** 68.80.	**11.** 28.87 ft.	**14.** 123.6 ft.
3. 162.6.	**6.** 26.84.	**9.** 45°.	**12.** 428.4 ft.	

Exercise 10. Page 21

1. 40.40.	**4.** 33.63.	**7.** 41°.	**10.** 57.74 ft.	**13.** 26.11 ft.
2. 61.77.	**5.** 55.50.	**8.** 60°.	**11.** 1369 ft.	
3. 101.2.	**6.** 339.4.	**9.** 22.65 ft.	**12.** 91.64 ft.	

Exercise 11. Page 22

1. 49.50. **2.** 54.87. **3.** 80.62. **4.** 64.60. **5.** 81.19. **6.** 152.8. **7.** 64°. **8.** 28°. **9.** 65°. **10.** 45°. **11.** 1113 ft. **12.** 13.69 mi. **13.** 19.82 mi. **14.** 267.0 ft. **15.** 57.51 ft. **16.** 17.23 in.

Exercise 12. Page 23

3. $\tan x$. **4.** $\sec x$. **5.** $\sec x$. **6.** $\csc x$. **7.** $\cot x$. **8.** $\csc x$. **16.** 18°. **35.** $r \sin x$. **36.** $a = cm$; $b = c\sqrt{1-m^2}$. **37.** $a = bm$; $c = b\sqrt{m^2+1}$.

Exercise 13. Page 26

2. 0. **3.** 1. **4.** ∞. **5.** 0. **6.** The tangent. **7.** No. **8.** No. **9.** 45°. **10.** 0.6462; 0.7631. **11.** 0.3680. **12.** 2.7173. **13.** 2.3109. **14.** 0.5373. **15.** 6°. **16.** 24°. **17.** 44°. **18.** 26°. **19.** 37°. **20.** 46°. **21.** 6°. **22.** 13°. **23.** 22°. **24.** 14°. **25.** 19°. **26.** 48°. **27.** 34°. **28.** 40°. **29.** 54°. **30.** 30°. **31.** 16°. **32.** 37°. **33.** $\frac{1}{2}$.

Exercise 14. Page 29

1. 0.7547. **2.** 0.9004. **3.** 0.7545. **4.** 0.9015. **5.** 0.7538. **6.** 0.7545. **7.** 0.7428. **8.** 0.6563. **9.** 0.6693. **10.** 0.6567. **11.** 0.6700. **12.** 0.6700. **13.** 0.8708. **14.** 0.8708. **15.** 1.1483. **16.** 17.73. **17.** 32.16. **18.** 46.01. **19.** 53.47. **20.** 20.90. **21.** 25.27. **22.** 48.29. **23.** 66.48. **24.** 64.84. **25.** 69.38. **26.** 49.83. **27.** 94.35. **28.** 74.93. **29.** 88.35. **30.** 47° 56′. **31.** 19.70 ft.; 22.62 ft. **32.** 19.72 ft.; 22.61 ft. **33.** 120.5 ft. **34.** 71.77 ft.

Exercise 15. Page 30

1. 0.0087. **2.** 0.0070. **3.** 0.0698. **4.** 0.9973. **5.** 0.0787. **6.** 0.0715. **7.** 0.9972. **8.** 0.0769. **9.** 12.71. **10.** 13.62. **11.** 0.9972. **12.** 0.9974. **13.** 0.0767. **14.** 13.95. **15.** 0.0769. **16.** 1.0000. **17.** 0.0715. **18.** 143.2. **19.** 0.0052. **20.** 0.0734. **21.** 12.66 in.; 0.9970 in. **22.** 390 ft. **23.** 0.7477 in.; 9.530 in.

Exercise 16. Page 33

1. 0.4567. **2.** 0.6725. **3.** 0.8338. **4.** 0.9099. **5.** 0.8065. **6.** 0.7289. **7.** 0.4335. **8.** 0.5438. **9.** 0.6418. **10.** 0.9209. **11.** 1.2882. **12.** 2.5018. **13.** 3.1266. **14.** 12.1524. **15.** 15.3140. **16.** 10.4652. **17.** 8.7149. **18.** 7.2246. **19.** 6.5585. **20.** 6.0826. **21.** 39° 43′ 30″; 0.7691. **22.** 50° 16′ 30″; 0.6391. **23.** 71° 29′ 40″; 0.9483. **24.** 70° 45′ 30″; 0.3490. **25.** 79° 30′ 15″; 0.1852. **26.** 0.4305. **27.** 0.4313. **28.** 0.5410. **29.** 0.6646. **30.** 0.9045. **31.** 0.1990. **32.** 4.9550. **33.** 0.1490. **34.** 7.8279. **35.** 10.7389. **36.** 0.9808. **37.** 4.5787. **38.** 4.1525. **39.** 3.6108. **40.** 3.3502. **41.** 31° 30′. **42.** 35° 15′. **43.** 41° 18′ 30″. **44.** 44° 36′ 30″. **45.** 38° 15′. **46.** 39° 30′. **47.** 17° 45′. **48.** 44° 38′ 30″. **49.** 69° 15′. **50.** 78° 8′ 30″. **51.** 78° 8′ 15″. **52.** 14° 45′. **53.** 0.7658. **54.** 0.6438. **55.** 0.5639. **56.** 33° 10′ 15″; 1.5298. **57.** 31° 8′ 30″; 0.6042.

Exercise 17. Page 37

1. $A = 36° 52'$, $B = 53° 8'$, $c = 5$.
2. $A = 32° 35'$, $B = 57° 25'$, $b = 10.95$.
3. $B = 77° 43'$, $b = 24.34$, $c = 24.93$.
4. $A = 46° 42'$, $b = 9.801$, $c = 14.29$.
5. $B = 52° 18'$, $a = 15.90$, $b = 20.57$.
6. $A = 65° 48'$, $a = 127.7$, $b = 57.39$.
7. $A = 34° 18'$, $B = 55° 42'$, $a = 12.96$.
8. $A = 43° 33'$, $B = 46° 27'$, $a = 93.14$.
9. $B = 57° 46'$, $a = 26.73$, $c = 50.12$.
10. $A = 43° 49'$, $a = 191.9$, $c = 277.2$.
11. $A = 68° 43'$, $B = 21° 17'$, $c = 102.0$.
12. $A = 3° 20'$, $B = 86° 40'$, $b = 102.8$.
13. $A = 84° 52'$, $b = 0.2802$, $c = 3.133$.
14. $A = 70° 48'$, $B = 19° 12'$, $b = 5.916$.
15. $B = 51° 31'$, $a = 35.47$, $b = 44.62$.
16. $A = 22° 37'$, $B = 67° 23'$, $a = 5$, $c = 13$.
17. $A = 53° 8'$, $B = 36° 52'$, $a = 40$, $c = 50$.
18. $A = 22° 37'$, $B = 67° 23'$, $a = 12.5$, $c = 32.5$.
19. $B = 54° 49' 30''$, $b = 3.547$, $c = 4.340$.
20. $B = 47° 47' 30''$, $b = 6.284$, $c = 8.485$.
21. $A = 60° 41' 30''$, $b = 3.593$, $c = 7.339$.
22. $A = 53° 39' 30''$, $b = 5.812$, $c = 9.808$.
23. $B = 60° 17' 30''$, $a = 3.370$, $b = 5.906$.
24. $B = 55° 39' 30''$, $a = 203.08$, $b = 297.25$.
25. $B = 48° 49' 20''$, $a = 218.68$, $c = 332.14$.
26. $B = 64.5°$, $b = 100.6$, $c = 111.5$.
27. $B = 65.5°$, $a = 10.37$, $b = 22.75$.
28. $B = 57.45°$, $a = 21.52$, $b = 33.72$.
29. $B = 34.49°$, $a = 65.94$, $b = 45.30$.
30. $B = 26.54°$, $a = 67.10$, $b = 33.51$.
31. $A = 39.41°$, $b = 54.77$, $c = 70.88$.
32. $B = 21.75°$, $a = 225.6$, $c = 242.8$.
33. 29.20 in.
34. 23.73 in.
35. 42.25 in.
36. 54.26 in.
37. 43.30 in.
38. 60.05 in.
39. 56° 18′ 36″, 33° 41′ 24″.
40. $A = 41° 24' 30''$, $B = 48° 35' 30''$.
41. 13.26 ft.
42. 16.82 in.; 18.50 in.
43. 12.42 ft.
44. 66.89 ft.
45. 9° 35′ 40″.

Exercise 18. Page 41

1. 5. **2.** 2. **3.** 4. **4.** 4. **5.** 6. **6.** 7. **7.** 8. **8.** 5. **9.** 6. **10.** 4. **11.** 3. **12.** 2. **13.** 3. **14.** 3. **15.** 4. **16.** 2. **17.** 3. **18.** 5. **19.** 6. **20.** −1.

21. −2; −3; −4.
22. 1 and 2; 2 and 3; 3 and 4; 4 and 5; 5 and 6; 8 and 9.
23. −2 and −1; −3 and −2; −4 and −3; −1 and 0; −2 and −1; −3 and −2.
24. 1; 2; 3; 6; 9; 10; −2; −4; −5; −6; −7; −8.
25. 1; 4; 6; 7; 8; −1; −2; −3; −4; −5; −6; −7.
26. 0; −4; −5; 7; 8.
27. 1 and 2.
28. 1 and 2.
29. 1 and 2.
30. 1 and 2.
31. 2 and 3.
32. 2 and 3.
33. 2 and 3.
34. 2 and 3.
35. 3 and 4.
36. 3 and 4.
37. 3 and 4.
38. 3 and 4.
39. 5 and 6.
40. 6 and 7.
41. 6 and 7.
42. 7 and 8.

Exercise 19. Page 45

1. 1. **2.** 1. **3.** 2. **4.** 0. **5.** 3.
6. 3. **7.** 2. **8.** 1. **9.** 0. **10.** 4.
11. −1. **12.** −2. **13.** −1. **14.** −1. **15.** −3.
16. −4. **17.** −3. **18.** −5. **19.** −1. **20.** −2.
21. 1.58681. **22.** 0.58681. **23.** 2.58681. **24.** 4.58681. **25.** 5.58681.

26. 7.58681.
27. $\bar{1}.58681$.
28. $\bar{2}.58681$.
29. $\bar{4}.58681$.
30. 3.67724.
31. 0.67724.
32. 4.67724.
33. 7.67724.
34. $\bar{2}.67724$.
35. $\bar{5}.67724$.
36. 0.40603.
37. 1.40603.
38. $\bar{1}.40603$.
39. $\bar{3}.40603$.
40. 4.40603.
41. 7.40603.
42. 0.39794.
43. $\bar{1}.39794$.
44. $\bar{1}.39794$.
45. $\bar{2}.39794$.
46. 4.39794.
47. 7.39794.

Exercise 20. Page 47

1. 0.30103.
2. 1.30103.
3. 2.30103.
4. $\bar{3}.30103$.
5. 3.32222.
6. 3.33244.
7. 3.33365.
8. 0.33365.
9. 3.54220.
10. 3.64953.
11. 3.74671.
12. 3.84553.
13. 3.72304.
14. 1.83556.
15. 0.89905.
16. 2.92158.
17. $\bar{1}.84510$.
18. $\bar{1}.87506$.
19. $\bar{1}.87852$.
20. $\bar{1}.87892$.
21. $\bar{2}.40654$.
22. $\bar{3}.55630$.
23. $\bar{4}.95424$.
24. $\bar{2}.25042$.
25. 4.09132.
26. 4.09150.
27. 4.09157.
28. 2.09157.
29. 2.37037.
30. 1.61624.
31. 1.75037.
32. $\bar{1}.61576$.
33. 5.51409.
34. 2.56155.
35. 7.82948.
36. 17.72562.
37. 9.19605.
38. 5.26893.
39. 2.51989.
40. 3.20732.
41. 4.86198.
42. 0.48124.
43. 0.95424.
44. 0.90309.
45. 4.22472.
46. 2.87595.
47. 5.32328.
48. 12.70040.
49. 19.58460.
50. 0.15052.
51. 1.65052.
52. 1.17969.
53. 0.46458.
54. 0.64167.
55. 1.08030.
56. 2.16224.
57. 0.79034.
58. 1.14477.
59. 0.54254.
60. 0.99155.
61. 2.00072.
62. 0.75343.
63. 1.19855.

Exercise 21. Page 49

1. 3.
2. 3000.
3. 0.003.
4. 304.5.
5. 37,020.
6. 46.
7. 467.5.
8. 0.000056.
9. 5505.
10. 0.05795.
11. 0.0006095.
12. 0.66.
13. 6.695.
14. 7.6.
15. 7,805,000,000.
16. 79,950,000.
17. 1.7102.
18. 27.005.
19. 370.15.
20. 0.38055.
21. 0.0043142.
22. 43,144.
23. 4.3646.
24. 0.049074.
25. 594,640,000.
26. 0.00067555.
27. 6846.5.
28. 685.55.
29. 77,553.
30. 785.65.
31. 7917.3.
32. 8.5552.
33. 875.18.
34. 2.
35. 3.45591; 3.45864.
36. 2955.
37. 0.0066062.
38. 0.65163.
39. 91.226.
40. 53,159,000.
41. 0.000010745.
42. 5.72784; 534,360.
43. 353,780.
44. 7.2388.
45. 107.
46. 25,459.
47. 16,693,000.
48. 129.66.
49. 4.9341.

Exercise 22. Page 50

1. 10.
2. 24.
3. 15.
4. 35.
5. 8.
6. 21.
7. 12.
8. 18.
9. 56.
10. 18.
11. 100.
12. 2400.
13. 1500.
14. 3500.
15. 8000.
16. 21,000.
17. 12,000.
18. 18,000.
19. 560,000.
20. 180,000.
21. 1034.6.
22. 2192.3.
23. 13.31.
24. 20.265.
25. 603.9.
26. 1282.8.
27. 184,670.
28. 11,099.
29. 1609.9.
30. 17,458.
31. 18.212 in.
32. 113.04 ft.
33. 210.
34. 945.
35. 5005.
36. 38,645.
37. 627,400.
38. 276.67.

Exercise 23. Page 51

1. 7.68964.
2. 3.68964.
3. $\bar{7}.68964$.
4. $\bar{3}.09497$.
5. 0.00000.
6. $\bar{1}.99999$.
7. $\bar{4}.03939$.
8. 2.00010.
9. 1.99999.
10. 0.00000.
11. 1,248,000.
12. 124.8.
13. 0.1248.
14. 0.0001248.
15. 0.0043707.
16. 0.11422.
17. 0.0000003125.
18. 0.25121.
19. 0.02240.
20. 0.00015725.
21. 1.3020.
22. 38.079.
23. 3309.6.
24. 452.27.
25. 22.936.
26. 34.108.
27. 16.51.

Exercise 24. Page 53

1. 1.97519.
2. $\bar{3}.66078$.
3. $\bar{1}.68618$.
4. $\bar{3}.70404$.
5. $\bar{5}.00000$.
6. $\bar{9}.70000$.
7. $\bar{7}.00000$.
8. $\bar{7}.00000$.
9. $\bar{3}.76439$.
10. 2.00000.
11. 2.90000.
12. 6.90000.
13. 3.89100.
14. $\bar{2}.00000$.
15. $\bar{2}.11220$.
16. $\bar{2}.00286$.
17. 1.71172.
18. 5.
19. 5.
20. 3.
21. 4.
22. 3.
23. 5.
24. 3.
25. 5.
26. 84.
27. 82.002.
28. 76.
29. 35.6.
30. 73.002.
31. 92.
32. 105.
33. 63.
34. 77.
35. 0.0129.
36. 1290.
37. 0.00999.
38. 0.0709.
39. 0.0204.
40. 0.065.
41. 0.48001.
42. 2.143.
43. 0.4667.
44. 0.004667.
45. 1.913.
46. 1.123.
47. 12.86.
48. 5.184.
49. 60.87.
50. 0.6527.
51. 20.
52. 50.
53. 700.
54. 800.
55. 9000.
56. 11,000.
57. 120,000.
58. 0.01.
59. 87.11; 2.

Exercise 25. Page 54

1. $\bar{2}.60206$.
2. $\bar{3}.88606$.
3. $\bar{2}.56225$.
4. $\bar{1}.23433$.
5. $\bar{4}.42585$.
6. $\bar{3}.36927$.
7. $\bar{2}.28727$.
8. $\bar{1}.14188$.
9. 0.30103.
10. 0.14267.
11. 1.08092.
12. 2.13906.
13. $\bar{1}.52187$.
14. $\bar{2}.20698$.
15. 3.22185.
16. 4.15490.
17. 1.
18. 0.1.
19. 0.
20. 1.

Exercise 26. Page 55

1. 1.
2. 6.
3. 3.
4. 0.5.
5. 1.
6. 2.
7. 0.11111.
8. 0.44272.
9. 1.7833.
10. 1000.
11. 0.092.
12. 1.8.
13. 0.01.
14. 0.21.
15. 6.1649.
16. 0.42742.
17. 1.4179.
18. 0.031169.
19. 40.464.
20. 0.14621.
21. 2893.2.
22. 105.47.
23. 3,013,400.
24. 0.081528.
25. 232.24.
26. 0.0000007237
27. 103.33.

Exercise 27. Page 56

1. 4.
2. 8.
3. 32.
4. 1024.
5. 80.998.
6. 729.98.
7. 64.
8. 125.
9. 1.
10. 40,355,000.
11. 4,782,800.
12. 16,777,000.
13. 19,486,000.
14. 11,391,000.
15. 11.391.
16. 83,522.
17. 15,625.
18. 6,103,600,000.
19. 15,625.
20. 244,140,000.

21. 16,413,000,000,000,000.
22. 7,700,500.
23. 31,137,000,000.
24. 292,360,000,000,000.
25. 2.1435.
26. 180.11.
27. 0.000000000001.
28. 0.00000002048.
29. 0.05765.
30. 0.00000011765.
31. 0.018741.
32. 154.85.
33. 157.5.
34. 41,961.
35. 2.0727.
36. 0.0019720.
37. 0.023551.
38. 0.00015228.
39. 0.0000075624.
40. 0.00000012603.
41. 9.8696; 31.006.
42. 21.991; 153.94, 3053.6.

Exercise 28. Page 57

1. 1.4142.
2. 1.71.
3. 1.3205.
4. 1.2394.
5. 1.1487.
6. 2.2795.
7. 5.6569.
8. 3.0403.
9. 3.3166.
10. 1.4422.
11. 2.802.
12. 1.2023.
13. 0.54773.
14. 0.3684.
15. 0.067405.
16. 0.064491.
17. 20.729.
18. 1.9733.
19. 3.9095.
20. 0.0028827.
21. 1.7725; 1.4645.
22. 1.3313; 2.1450; 5.5684; 0.42378; 0.40020; 0.79537.

Exercise 29. Page 59

1. $x = 3$.
2. $x = 4$.
3. $x = 4$.
4. $x = 4$.
5. $x = 3$.
6. $x = 4.2479$.
7. $x = 3.9300$.
8. $x = 4.2920$.
9. $x = 5.6610$.
10. $x = 3.0499$.
11. $x = 3$.
12. $x = 3.3219$.
13. $x = 0.087515$.
14. $x = 4.4190$.
15. $x = -0.047954$.
16. $x = 3, y = 1$.
17. $x = 5, y = 1$
18. $x = 1, y = 1$.
19. $x = 2, y = 2$.
20. $x = 3, y = 2$.
21. $x = 2, y = 2$.
22. $x = \dfrac{\log a - \log p}{\log (1 + r)}$.
23. $x = \dfrac{\log r + \log l - \log a}{\log r}$.
24. $x = 1, -3$.
25. $x = \dfrac{\log a - \log p}{\log a (1 + rt)}$.
26. $x = \dfrac{\log s + \log (r - 1)}{\log r}$.
27. $x = 2, -1$.
28. 0.062457.
29. 3.1389.
30. 0.036161.
31. 0.03475.
32. 6.
33. $\dfrac{\log b}{\log a}$.
34. $\dfrac{\log n}{\log 5}$.
35. 2; 7.2730; 2.0009; 2.0043.
36. 1; $\dfrac{\log a}{\log b}$; 1; 3; 4.
37. $x = \dfrac{\log b}{\log a - \log b}$.
38. -1.

Exercise 30. Page 62

1. 9.65705 – 10.
2. 9.97015 – 10.
3. 9.90796 – 10.
4. 9.82551 – 10.
5. 10.57195 – 10.
6. 9.32747 – 10.
7. 10.57195 – 10.
8. 9.32747 – 10.
9. 9.20613 – 10.
10. 9.99526 – 10.
11. 9.14412 – 10.
12. 9.14412 – 10.
13. 8.89464 – 10.
14. 9.99651 – 10.
15. 9.23510 – 10.
16. 9.87099 – 10.
17. 9.68826 – 10.
18. 10.10706 – 10.
19. 9.55763 – 10.
20. 9.96966 – 10.
21. 9.98436 – 10.
22. 9.42095 – 10.
23. 9.48632 – 10.
24. 9.68916 – 10.
25. 9.95340 – 10.
26. 11.13737 – 10.
27. 9.74766 – 10.
28. 9.66368 – 10.
29. 10.17675 – 10.
30. 9.82332 – 10.
31. 6.51059 – 10.
32. 8.25667 – 10.
33. 6.79257 – 10.
34. 8.56813 – 10.
35. 7.45643 – 10.
36. 8.15611 – 10.
37. 8.11503 – 10.
38. 8.00469 – 10.
39. 8.24915 – 10.
40. 8.24915 – 10.
41. 8.63254 – 10.
42. 8.63205 – 10.
43. 9.32507 – 10.
44. 9.32507 – 10.
45. 10.39604 – 10
46. 7° 30′.
47. 32° 21′.
48. 58° 27′.

49. 85° 30′.	**55.** 63° 41′ 23″.	**61.** 49° 34′ 12″.	**67.** 57° 4$\frac{2}{7}$″.
50. 4° 30′.	**56.** 77° 6′.	**62.** 51° 47′ 36″.	**68.** 49° 25′ 7″.
51. 31° 33′.	**57.** 79°.	**63.** 37° 8′ 48″.	**69.** 38° 22′ 30″.
52. 58° 35′.	**58.** 70°.	**64.** 50° 48′ 15″.	**70.** 2° 3′ 30″.
53. 50° 32′.	**59.** 20° 13′ 30″.	**65.** 8° 49′ 30″.	**71.** 89° 49′ 10″.
54. 39° 2′.	**60.** 32° 22′ 15″.	**66.** 8° 46′ 30″.	

Exercise 31. Page 67

1. $A = 30°$,	$B = 60°$,	$b = 10.39$,	$S = 31.18$.
2. $B = 30°$,	$a = 6.928$,	$c = 8$,	$S = 13.86$.
3. $B = 60°$,	$b = 5.196$,	$c = 6$,	$S = 7.794$.
4. $A = 45°$,	$B = 45°$,	$c = 5.657$,	$S = 8$.
5. $A = 43° 47'$,	$B = 46° 13'$,	$b = 2.086$,	$S = 2.086$.
6. $B = 66° 30'$.	$a = 250$,	$b = 575$,	$S = 71{,}880$.
7. $B = 61° 55'$,	$a = 1073$,	$b = 2012$,	$S = 1{,}079{,}500$.
8. $B = 50° 26'$,	$a = 45.96$,	$b = 55.62$,	$S = 1278$.
9. $B = 54°$,	$a = 0.5878$,	$b = 0.8090$,	$S = 0.2378$.
10. $A = 68° 13'$,	$a = 185.7$,	$b = 74.22$,	$S = 6892$.
11. $A = 13° 35'$,	$a = 21.94$,	$b = 90.79$,	$S = 995.8$.
12. $B = 85° 25'$,	$b = 7946$,	$c = 7972$,	$S = 2{,}531{,}000$.
13. $B = 53° 16'$,	$b = 65.03$,	$c = 81.14$,	$S = 1578$.
14. $B = 4°$,	$b = 0.0005594$,	$c = 0.00802$,	$S = 0.000002238$.
15. $A = 46° 12'$,	$a = 53.12$,	$c = 73.60$,	$S = 1353$.
16. $A = 86° 22'$,	$a = 31.50$,	$c = 31.56$,	$S = 31.50$.
17. $A = 13° 41'$,	$b = 4075$,	$c = 4194$,	$S = 2{,}021{,}000$.
18. $A = 21° 8'$,	$b = 188.9$,	$c = 202.5$,	$S = 6893$.
19. $A = 44° 35'$,	$b = 2.221$,	$c = 3.119$,	$S = 2.431$.
20. $B = 52° 4'$,	$a = 3.118$,	$c = 5.071$,	$S = 6.235$.
21. $A = 31° 24'$,	$B = 58° 36'$,	$b = 7333$,	$S = 16{,}410{,}000$.
22. $A = 56° 3'$,	$B = 33° 57'$,	$b = 48.32$,	$S = 1734$.
23. $A = 65° 14'$,	$B = 24° 46'$,	$b = 3.917$,	$S = 16.63$.
24. $A = 53° 15'$,	$B = 36° 45'$,	$a = 1758$,	$S = 1{,}154{,}000$.
25. $A = 53° 31'$,	$B = 36° 29'$;	$a = 24.68$,	$S = 225.2$.
26. $A = 63°$,	$B = 27°$,	$c = 43$,	$S = 373.9$.
27. $A = 4° 42'$,	$B = 85° 18'$,	$c = 15$,	$S = 9.187$.
28. $A = 81° 30'$,	$B = 8° 30'$,	$c = 419.9$,	$S = 12{,}890$.
29. $A = 38° 59'$,	$B = 51° 1'$,	$c = 21.76$,	$S = 115.8$.
30. $A = 1° 22'$,	$B = 88° 38'$,	$b = 91.89$,	$S = 100.6$.
31. $A = 39° 48'$,	$B = 50° 12'$,	$c = 7.811$,	$S = 15$.
32. $A = 30' 12''$,	$B = 89° 29' 48''$,	$b = 70$,	$S = 21.53$.
33. $A = 43° 20'$,	$B = 46° 40'$,	$a = 1.189$,	$S = 0.7488$.
34. $B = 71° 46'$,	$b = 21.25$,	$c = 22.37$,	$S = 74.37$.
35. $B = 60° 52'$,	$a = 6.688$,	$c = 13.74$,	$S = 40.13$.
36. $B = 20° 6'$,	$a = 63.86$,	$b = 23.37$,	$S = 746.15$.
37. $A = 45° 56'$,	$a = 19.40$,	$b = 18.78$,	$S = 182.15$.
38. $A = 41° 11'$,	$b = 53.72$,	$c = 71.38$,	$S = 1262.4$.
39. $A = 55° 16'$,	$a = 12.98$,	$c = 15.80$,	$S = 58.42$.
40. $A = 3° 56'$,	$a = 0.5805$,	$b = 8.442$,	$S = 2.450$.

41. $S = \frac{1}{2}c^2 \sin A \cos A.$
42. $S = \frac{1}{2}a^2 \cot A.$
43. $S = \frac{1}{2}b^2 \tan A.$
44. $S = \frac{1}{2}a\sqrt{c^2 - a^2}.$
45. $A = 40°\,45'\,48''$, $B = 49°\,14'\,12''$, $b = 11.6$, $c = 15.315.$
46. $A = 55°\,13'\,20''$, $B = 34°\,46'\,40''$, $a = 7.2$, $c = 8.766.$
47. $B = 61°$, $a = 3.647$, $b = 6.58$, $c = 7.523.$
48. $A = 27°\,2'\,30''$, $B = 62°\,57'\,30''$, $a = 10.002$, $b = 19.595.$
49. 19° 28′ 17″; 70° 31′ 43″.
50. 3112 mi.; 19,553 mi.
51. 15.498 mi.
52. Between 1° 15′ 30″ and 1°19′ 10″.
53. 212.1 ft.
54. 732.2 ft.
55. 3270 ft.
56. 37.3 ft.
57. 1° 25′ 56″.
58. 59° 44′ 35″.
59. 95.34 ft.
60. 23° 50′ 40″.
61. 36° 1′ 42″.
62. 69° 26′ 38″.
63. 7.071 mi.; 7.071 mi.
64. 19.05 ft.
65. 20.88 ft.
66. 56.65 ft.
67. 685.9 ft.
68. 5.657 ft.
69. 136.6 ft.
70. 140 ft.
71. 84.74 ft.

Exercise 32. Page 71

1. $C = 2(90° - A)$, $c = 2a\cos A$, $h = a\sin A.$
2. $A = 90° - \frac{1}{2}C$, $c = 2a\cos A$, $h = a\sin A.$
3. $C = 2(90° - A)$, $a = \dfrac{c}{2\cos A}$, $h = a\sin A.$
4. $A = 90° - \frac{1}{2}C$, $a = \dfrac{c}{2\cos A}$, $h = a\sin A.$
5. $C = 2(90° - A)$, $a = \dfrac{h}{\sin A}$, $c = 2a\cos A.$
6. $A = 90° - \frac{1}{2}C$, $a = \dfrac{h}{\sin A}$, $c = 2a\cos A.$
7. $\sin A = \dfrac{h}{a}$, $C = 2(90° - A)$, $c = 2a\cos A.$
8. $\tan A = \dfrac{2h}{c}$, $C = 2(90° - A)$, $a = \dfrac{h}{\sin A}.$
9. $A = 67°\,22'\,50''$, $C = 45°\,14'\,20''$, $h = 13.2.$
10. $c = 0.21943$, $h = 0.27384$, $S = 0.03004.$
11. $a = 2.055$, $h = 1.6852$, $S = 1.9819.$
12. $a = 7.706$, $c = 3.6676$, $S = 13.725.$
13. $A = 25°\,27'\,47''$, $C = 129°\,4'\,26''$, $a = 81.41$, $h = 35.$
14. $A = 81°\,12'\,9''$, $C = 17°\,35'\,42''$, $a \doteq 17$, $c = 5.2.$
15. $c = 14.049$, $h = 26.649$, $S = 187.2.$
16. $S = a^2 \sin\frac{1}{2}C \cos\frac{1}{2}C.$
17. $S = a^2 \sin A \cos A.$
18. $S = h^2 \tan\frac{1}{2}C.$
19. 28.284 ft.; 4525.44 sq. ft.
20. 0.76536.
21. 94° 20′.
22. 2.7261.
23. 38° 56′ 33″.
24. 37.699 sq. in.
25. 0.8775.

Exercise 33. Page 72

1. $r = 1.618$, $h = 1.5388$, $S = 7.694.$
2. $h = 0.9848$, $p = 6.2514$, $S = 3.0782.$
3. $h = 19.754$, $c = 6.257$, $S = 1236.$
4. $r = 1.0824$, $c = 0.82842$, $S = 3.3137.$
5. $r = 2.5933$, $h = 2.4882$, $c = 1.4615.$
6. $r = 1.5994$, $h = 1.441$, $p = 9.716.$
7. 0.51764 in.
8. $b = \dfrac{c}{2\cos\dfrac{90°}{n}}.$
9. 0.2238 sq. in.
10. 0.310 in.
11. 1.0235 in.
12. 0.062821; 6.2821.
13. 6.283.
14. 0.635 sq. in.

Exercise 34. Page 73

2. 29.76 sq. in.
3. 104.07 sq. ft.
4. 36.463 sq. in.
5. 20.284 in.
7. 37.319 ft.
8. 342.67 ft.
9. 36.602 ft.; 86.602 ft.
10. 120.03 ft.
11. 2.9101 mi.; 3.531 mi.
12. $11° 47''$; 49.206 ft.
13. $52° 35' 42''$.
14. $60° 36' 58''$.
15. 6.3509 in.
16. 20 in.
17. 7.7942 in.
18. $40° 7' 6''$.
19. $77° 8' 31''$.
20. 94.368 ft.; $25° 42' 58''$.
21. 24.652 ft.
22. 196.93 ft.
23. 220.8 ft.
24. 1915.3 ft.
25. 362.09 ft.
26. $59° 2' 10''$.
27. 14.772 in.; 15.595 in.
28. 73.21 ft.
29. $25° 36' 9''$.
30. 26.613 in.
31. 7.5 ft.
32. $59° 58' 54''$; 173.08 ft.
33. 7.2917 ft.
34. 19.051.
35. 1.732 in.
36. 2675.8 mi.
37. 25.775 ft.; 19.45 ft.
38. 10.941 ft.; 20.141 ft.
39. 55.406 ft.
40. Between 131' and 132'.
41. $43° 18' 48''$.
42. 2.6068 in.
43. 14.542 in.; 26.87 in.
44. 6470.36 ft.

Exercise 35. Page 80

29. 10.
30. 15.
31. 13.
32. $2\frac{1}{2}$.
33. $1\frac{1}{4}$.
34. $3\frac{3}{4}$.
35. 3.
36. 5.
37. 0.
38. 7.
39. 5.
40. 15.
41. 5.10.
42. 5.10.
43. 8.24.
44. 4.24.
45. $28\frac{1}{3}$ in.
46. 9.43 in.
47. 2.
48. $3\sqrt{3}$.
49. $\frac{3}{4}\sqrt{3}$.
50. Yes.
51. Octagon; 2.829.

Exercise 36. Page 84

16. I.
17. I.
18. II.
19. II.
20. III.
21. IV.
22. I.
23. II.
24. III.
25. IV.
26. I.
27. II.
28. III.
29. On OY'.
30. On OX.
61. $\frac{1}{3}\sqrt{6}$; $\frac{1}{3}\sqrt{3}$; $\sqrt{3}$; $\frac{1}{2}\sqrt{6}$.
62. 90°.
63. 60°.
64. $\sin = \frac{1}{2}\sqrt{2}$; $\cos = -\frac{1}{2}\sqrt{2}$; $\tan = -1$; $\csc = \sqrt{2}$; $\sec = -\sqrt{2}$; $\cot = -1$.
65. $\sin = 0$; $\cos = -1$; $\tan = 0$; $\csc = \infty$; $\sec = -1$; $\cot = \infty$.

Exercise 37. Page 88

52. 2; one in Quadrant I, one in Quadrant II.
53. 4; two in Quadrant I, two in Quadrant IV.
54. 2; 1; 1; 1; 1.
55. Between 90° and 270°; between 0° and 90° or between 180° and 270°; between 0° and 90° or between 270° and 360°; between 180° and 360°.
57. 1; 0; 0; ∞; 1; ∞; 1; 0.
59. III; II.
60. 40; 20.
61. 0.
62. 0.
63. 0.
64. $a^2 - b^2 + 4ab$.
65. $-2(a^2 + b^2)$.
66. 0.
67. 0.
76. 30°; 150°; 390°; 510°.
77. 30°; 330°; 390°; 690°.
78. 60°; 120°; 420°; 480°.
79. 60°; 300°; 420°; 660°.
80. 30°; 210°; 390°; 570°.
81. 60°; 240°; 420°; 600°.
82. 210°; 330°.
83. 120°; 240°.
84. 225°; 315°.
85. 135°; 225°.
86. 135°; 315°.
87. 135°; 315°.

Exercise 38. Page 91

1. $\sin 10°$.
2. $-\cos 20°$.
3. $-\tan 32°$.
4. $-\cot 24°$.
5. $\sin 0°$.
6. $-\tan 0°$.
7. $-\sin 20°$.
8. $-\cos 45°$.
9. $\tan 78°$.
10. $\cot 82°$.
11. $-\sin 85°$.
12. $-\sin 15°$.
13. $-\tan 78°$.
14. $-\tan 35°$
15. $\cos 70°$.
16. $\cos 10°$.
17. $-\cot 65°$.
18. $-\cot 13°$.
19. $-\sin 0°$.
20. $\cos 0°$.
21. $\sin 31° 50'$.
22. $-\cos 12° 20'$.
23. $\tan 85° 30'$.
24. $-\cot 72° 20'$.
25. $-\sin 7° 10' 3''$.
26. $\cos 85° 54' 46''$.
27. $-\tan 37° 51' 45''$.
28. $\cot 15° 10' 3''$.
29. $\sin 32.25°$.
30. $-\cos 52.25°$.

Exercise 39. Page 93

1. $\cos 10°$.
2. $\cos 30°$.
3. $\cos 20°$.
4. $\cos 40°$.
5. $-\sin 5°$.
6. $-\sin 7°$.
7. $-\sin 21°$.
8. $-\sin 37°$.
9. $-\cot 1°$.
10. $-\cot 9°$.
11. $-\cot 29°$.
12. $-\cot 39°$.
13. $-\tan 4° 1'$.
14. $-\tan 7° 2'$.
15. $-\tan 8° 3'$.
16. $-\tan 9° 9'$.
17. $-\sin 3°$.
18. $-\sin 9°$.
19. $-\sin 86°$.
20. $\cos 75°$.
21. $\cos 87°$.
22. $-\sin 5°$.
23. $\tan 80°$.
24. $\tan 30°$.
25. $-\tan 20°$.
26. $-\cot 1.5°$.
27. $-\cot 7.8°$.
28. $-\cot 9.1°$.
29. 0.0262.
30. -0.8013.
31. -0.7729.
32. 0.5040.
33. -0.1304.
34. 0.8686.
35. 0.1357.
36. -0.1354.
37. $9.89947 - 10$.
38. $-(9.83861 - 10)$.
39. $-(9.79916 - 10)$.
40. $-(10.52286 - 10)$.
41. $-(9.91969 - 10)$.
42. $9.92401 - 10$.
43. $10.14753 - 10$.
44. $-(9.82489 - 10)$.
46. $225°$; $315°$; $585°$; $675°$

Exercise 40. Page 95

6. $\sin x = \pm \dfrac{1}{\sqrt{\cot^2 x + 1}}$.
7. $\cos x = \pm \dfrac{1}{\sqrt{\tan^2 x + 1}}$.
8. $\sec x = \pm \dfrac{1}{\sqrt{1 - \sin^2 x}}$.
9. $\csc x = \pm \dfrac{1}{\sqrt{1 - \cos^2 x}}$.
19. $45°$.
20. $30°$.
21. $60°$.
22. $45°$.
23. $45°$.
24. $45°$.
25. $60°$.
26. $45°$.
27. $60°$.
28. $60°$ or $180°$.
29. $45°$.
30. $30°$.
31. $45°$.
32. $\frac{1}{5}\sqrt{5}$; $\frac{2}{5}\sqrt{5}$.
33. $\frac{1}{4}\sqrt{15}$; $\sqrt{15}$
34. $\frac{1}{5}$; 5.
35. $\sin x = \frac{2}{5}\sqrt{5}$, $\cos x = \frac{1}{5}\sqrt{5}$, $\tan x = 2$; $\csc x = \frac{1}{2}\sqrt{5}$, $\sec x = \sqrt{5}$, $\cot x = \frac{1}{2}$.
36. $\frac{4}{17}\sqrt{17}$; $\frac{1}{17}\sqrt{17}$.
37. $\frac{9}{41}$; $\frac{40}{41}$.
38. When $x = 0°$.
39. $0°$ or $180°$.
40. $38° 10'$.
41. $45°$ or $225°$.
42. $45°$, $135°$, $225°$, or $315°$.
43. $45°$ or $225°$.
44. $0°$ or $60°$.
45. $270°$ or $30°$.
46. $30°$ or $150°$.
47. $45°$, $135°$, $225°$, or $315°$.
48. $60°$.
53. $\cos A = \frac{1}{3}\sqrt{5}$, $\tan A = \frac{2}{5}\sqrt{5}$, $\csc A = \frac{3}{2}$, $\sec A = \frac{3}{5}\sqrt{5}$, $\cot A = \frac{1}{2}\sqrt{5}$.
54. $\sin A = \frac{1}{4}\sqrt{7}$, $\tan A = \frac{1}{3}\sqrt{7}$, $\csc A = \frac{4}{7}\sqrt{7}$, $\sec A = \frac{4}{3}$, $\cot A = \frac{3}{7}\sqrt{7}$.
55. $\sin A = \frac{3}{13}\sqrt{13}$, $\cos A = \frac{2}{13}\sqrt{13}$, $\csc A = \frac{1}{3}\sqrt{13}$, $\sec A = \frac{1}{2}\sqrt{13}$, $\cot A = \frac{2}{3}$.
56. $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$, $\csc A = \frac{5}{4}$, $\sec A = \frac{5}{3}$.
57. $\sin A = \frac{1}{3}\sqrt{5}$, $\cos A = \frac{2}{3}$, $\tan A = \frac{1}{2}\sqrt{5}$, $\csc A = \frac{3}{5}\sqrt{5}$, $\cot A = \frac{2}{5}\sqrt{5}$.
58. $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$, $\csc A = \frac{13}{12}$, $\sec A = \frac{13}{5}$, $\cot A = \frac{5}{12}$.
59. $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$, $\csc A = \frac{5}{4}$, $\sec A = \frac{5}{3}$, $\cot A = \frac{3}{4}$.
60. $\sin A = \frac{11}{61}$, $\tan A = \frac{11}{60}$, $\csc A = \frac{61}{11}$, $\sec A = \frac{61}{60}$, $\cot A = \frac{60}{11}$.

61. $\sin A = \frac{24}{25}$, $\tan A = \frac{24}{7}$, $\csc A = \frac{25}{24}$, $\sec A = \frac{25}{7}$, $\cot A = \frac{7}{24}$.
62. $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\csc A = \frac{5}{4}$, $\sec A = \frac{5}{3}$, $\cot A = \frac{3}{4}$.
63. $\sin A = \frac{1}{2}\sqrt{2}$, $\cos A = \frac{1}{2}\sqrt{2}$, $\tan A = 1$, $\csc A = \sqrt{2}$, $\sec A = \sqrt{2}$.
64. $\sin A = \frac{2}{5}\sqrt{5}$, $\cos A = \frac{1}{5}\sqrt{5}$, $\tan A = 2$, $\csc A = \frac{1}{2}\sqrt{5}$, $\sec A = \sqrt{5}$.
65. $\sin A = \frac{1}{2}\sqrt{3}$, $\cos A = \frac{1}{2}$, $\tan A = \sqrt{3}$, $\csc A = \frac{2}{3}\sqrt{3}$, $\cot A = \frac{1}{3}\sqrt{3}$.
66. $\sin A = \frac{1}{2}\sqrt{2}$, $\cos A = \frac{1}{2}\sqrt{2}$, $\tan A = 1$, $\sec A = \sqrt{2}$, $\cot A = 1$.
67. $\cos A = \sqrt{1-m^2}$, $\tan A = \dfrac{m}{\sqrt{1-m^2}}$, $\csc A = \dfrac{1}{m}$, $\sec A = \dfrac{1}{\sqrt{1-m^2}}$, $\cot A = \dfrac{\sqrt{1-m^2}}{m}$.
68. $\dfrac{2m}{1-m^2}$.
69. $\dfrac{m^2+n^2}{2mn}$.
70. $\cos 0° = 1$, $\tan 0° = 0$, $\csc 0° = \infty$, $\sec 0° = 1$, $\cot 0° = \infty$.
71. $\cos 90° = 0$, $\tan 90° = \infty$, $\csc 90° = 1$, $\sec 90° = \infty$, $\cot 90° = 0$.
72. $\sin 90° = 1$, $\cos 90° = 0$, $\csc 90° = 1$, $\sec 90° = \infty$, $\cot 90° = 0$.
73. $\sin 22° 30' = \dfrac{1}{\sqrt{4+2\sqrt{2}}}$, $\cos 22° 30' = \dfrac{1}{\sqrt{4-2\sqrt{2}}}$, $\tan 22° 30' = \sqrt{2}-1$,
$\csc 22° 30' = \sqrt{4+2\sqrt{2}}$, $\sec 22° 30' = \sqrt{4-2\sqrt{2}}$.
74. $\dfrac{1-\cos^2 A}{\cos A} + \dfrac{\cos^2 A}{1-\cos^2 A}$.

Exercise 41. Page 98

1. 0.25875.	5. 1.	9. 0.866.	13. 0.5.
2. 0.96575.	6. 0.	10. − 0.5.	14. − 0.866.
3. 0.96575.	7. 0.96575.	11. 0.707.	15. 0.25875.
4. 0.25875.	8. − 0.25875.	12. − 0.707.	16. − 0.96575

Exercise 42. Page 99

1. 0.268.	5. ∞.	9. − 1.732.	13. − 0.577.
2. 3.732.	6. 0.	10. − 0.577.	14. − 1.732.
3. 3.732.	7. − 3.732.	11. − 1.	15. − 0.268.
4. 0.268.	8. − 0.268.	12. − 1.	16. − 3.732.

Exercise 43. Page 102

1. $\frac{56}{65}$.
2. $\frac{16}{65}$.
3. $\frac{33}{65}$.
4. $\frac{63}{65}$.
5. $1\frac{23}{33}$.
6. $\frac{16}{63}$.
7. $\cos y$.
8. $\sin y$.
9. $\cot y$.
10. $\cos y$.
11. $\sin y$.
12. $-\sin y$.
13. $-\cos y$.
14. $-\cos y$.
15. $-\sin y$.
16. $\sin y$.
17. $\sin x$.
18. $-\cos x$.
19. $-\sin x$.
20. $-\cot x$.
21. $\tan x$.
22. $-\tan x$.
23. $\cot x$.
24. $-\sin y$.
25. $\frac{1}{2}\sqrt{2}(\cos y - \sin y)$.
26. $\frac{1}{2}\sqrt{2}(\cos y + \sin y)$.
27. $\dfrac{1-\tan y}{1+\tan y}$.
28. $\dfrac{\sqrt{3}\cot y - 1}{\cot y + \sqrt{3}}$.
29. $\dfrac{\frac{1}{3}\sqrt{3}\cot y + 1}{\cot y - \frac{1}{3}\sqrt{3}}$.
30. $\tan y$.
31. 0.8571; 0.2222.
32. 3.732; 0.268.
33. 1; $\frac{49}{71}$; 45°.
34. $x + y = 90°$, 270° in the three cases.
37. 135°, 405°.

Exercise 44. Page 103

5. 1. **7.** $-\frac{1}{2}$. **9.** 0.8492. **11.** -1.1776. **13.** $\frac{120}{169}$. **15.** $3\sin x - 4\sin^3 x$.
6. $\frac{1}{2}\sqrt{3}$. **8.** $\frac{7}{25}$. **10.** 0.5827. **12.** 1.7161. **14.** $\frac{120}{169}$. **16.** $4\cos^3 x - 3\cos x$.

Exercise 45. Page 104

1. 0.2588. **3.** 0.2679. **5.** 7.5928. **7.** 0.9239. **9.** 2.4142.
2. 0.9659. **4.** 3.7321. **6.** 0.3827. **8.** 0.4142. **10.** 5.0280.
11. 0.10051; 0.99493. **12.** 0.38730; 0.92196; 0.42009; 2.3805.

Exercise 46. Page 105

8. 0.
9. $\frac{1}{2}\sqrt{3}$.
15. $\dfrac{2}{\sin 2x}$.
16. $2\cot 2x$.
17. $\dfrac{\cos(x-y)}{\sin x \cos y}$.
18. $\dfrac{\cos(x+y)}{\sin x \cos y}$.
19. $\tan^2 x$.
20. $\dfrac{\cos(x-y)}{\cos x \cos y}$.
21. $\dfrac{\cos(x+y)}{\cos x \cos y}$.
22. $\dfrac{\cos(x-y)}{\sin x \sin y}$.
23. $\dfrac{\cos(x+y)}{\sin x \sin y}$.
24. $\tan x \tan y$.
27. $\frac{1}{7}$.

Exercise 47. Page 109

1. $a = b\sin A$; $b = a\sin B$; $a = b$; $\sin A = \sin B$.
4. 8 in.
5. 1000 ft.
6. 8.5450 in.; 4.2728 in.
7. 27.6498 in.
8. 9.1121 in.

Exercise 48. Page 110

1. $C = 123° 12'$, $b = 2051.5$, $c = 2362.6$.
2. $C = 55° 20'$, $b = 567.69$, $c = 663.99$.
3. $C = 35° 4'$, $b = 577.31$, $c = 468.93$.
4. $C = 25° 12'$, $b = 2276.6$, $c = 1573.9$.
5. $C = 47° 14'$, $a = 1340.6$, $b = 1113.8$.
6. $A = 108° 50'$, $a = 53.276$, $c = 47.324$.
7. $B = 56° 56'$, $b = 5685.9$, $c = 5357.5$.
8. $B = 77°$, $a = 630.77$, $c = 929.48$.
9. $a = 5$; $c = 9.659$.
10. $a = 7$; $b = 8.573$.
11. Sides, 600 ft. and 1039.2 ft.; altitude, 519.6 ft.
12. 855 : 1607.
13. 5.438; 6.857.
14. 15.588 in.
15. $AB = 59.564$ mi.; $AC = 54.285$ mi.
16. 4.1365 and 8.6416.
17. 6.1433 mi. and 8.7918 mi.
18. 6.4343 mi. and 5.7673 mi.
19. 8 and 5.4723.
20. 4.6064 mi.; 4.4494 mi.; 3.7733 mi.
21. 5.4709 mi.; 5.8013 mi.; 4.3111 mi.

Exercise 50. Page 115

1. Two. **3.** No solution. **5.** One. **7.** No solution
2. One. **4.** One. **6.** Two. **8.** One.
9. $B = 12° 13' 34''$, $C = 146° 15' 26''$, $c = 1272.1$.
10. $B = 57° 23' 40''$, $C = 2° 1' 20''$, $c = 0.38525$.
11. $B = 41° 12' 56''$, $C = 87° 38' 4''$, $c = 116.83$.
12. $A = 54° 31'$, $C = 47° 45'$. $c = 50.496$.
13. $B = 24° 57' 26''$, $C = 133° 48' 34''$, $c = 615.7$;
$B' = 155° 2' 34''$, $C' = 3° 43' 26''$, $c' = 55.414$.

14. $A = 51° 18' 27''$, $C = 98° 21' 33''$, $c = 43.098$;
$A' = 128° 41' 33''$, $C' = 20° 58' 27''$, $c' = 15.593$.

15. $A = 147° 27' 47''$, $B = 16° 43' 13''$, $a = 35.519$;
$A' = 0° 54' 13''$, $B' = 163° 16' 47''$, $a' = 1.0415$.

16. $B = 44° 1' 28''$, $C = 97° 44' 20''$, $c = 13.954$;
$B' = 135° 58' 32''$, $C' = 5° 47' 16''$, $c' = 1.4202$.

17. $B = 90°$, $C = 32° 22' 43''$, $c = 2.7901$.

18. 420. 19. 124.62. 20. 3.2096 in.

21. $AB = 3.8771$ in.; $BC = 2.3716$ in.; $CD = 3.7465$ in.; $AD = 6.1817$ in.

22. $C = 125° 6'$, $D = 93° 24'$; $AB = 4.3075$ in.; $BC = 3.1288$ in.; $CD = 5.431$ in.; $DE = 4.4186$ in.; $AE = 5.0522$ in.

Exercise 51. Page 117

2. $b = a \cos C + c \cos A$;
$a = b \cos C + c \cos B$;
$c = b \cos A$.

4. Impossible.

5. 5.

6. 7.655.

7. 7.

8. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$; 90°.

14. $AC = 8.499$ in.;
$BD = 3.1254$ in.

15. $BC = 5.9924$ in.;
$BD = 8.3556$ in.

16. $AB = 1.9246$ in.;
$CD = 4.4431$ in.;
$A = 109° 22' 30''$;
$B = 112° 13' 40''$;
$C = 88° 11' 40''$;
$D = 50° 12' 10''$.

17. 13.3157 in.

Exercise 52. Page 119

1. $\dfrac{a-b}{a+b} = \tan(A - 45°)$.

2. $\tan \frac{1}{2}(A - B) = 0$.

3. $a = b$.

4. $a + b = (a - b)\left(2 + \sqrt{3}\right)$.

11. $\dfrac{2 \sin A}{0} = \dfrac{\tan A}{0}$, or $\infty = \infty$.

12. $\dfrac{\sqrt{3}}{0} = \dfrac{\sqrt{3}}{0}$.

13. $\dfrac{\sqrt{3}}{0} = \infty \sqrt{3}$.

14. $\tan \frac{1}{2}(A - B) = 0$; $A = B$.

17. 5.

18. Sides AB, BC, AE; diagonal AD; angles B, CAD, DAE.

Exercise 53. Page 121

1. $A = 51° 15'$, $B = 56° 30'$, $c = 95.24$.

2. $B = 60° 45' 2''$, $C = 39° 14' 58''$, $a = 984.83$.

3. $A = 77° 12' 53''$, $B = 43° 30' 7''$, $c = 14.987$.

4. $B = 93° 28' 36''$, $C = 50° 38' 24''$, $a = 1.3131$.

5. $A = 132° 18' 27''$, $B = 14° 34' 24''$, $c = 0.6775$.

6. $A = 118° 55' 49''$, $C = 45° 41' 35''$, $b = 4.1554$.

7. $B = 65° 13' 51''$, $C = 28° 42' 5''$, $a = 3297.2$.

8. $A = 68° 29' 15''$, $B = 45° 24' 18''$, $c = 4449$.

9. $A = 117° 24' 32''$, $B = 32° 11' 28''$, $c = 31.431$.

10. $A = 2° 46' 8''$, $B = 1° 54' 42''$, $c = 81.066$.

11. $A = 116° 33' 54''$, $B = 26° 33' 54''$, $c = 140.87$.

12. $A = 6° 1' 55''$, $B = 108° 58' 5''$, $c = 862.5$.

13. $A = 45° 14' 20''$, $B = 17° 3' 40''$, $c = 510.02$.

14. $A = 41° 42' 33''$, $B = 32° 31' 15''$, $c = 9.0398$.

15. $A = 62° 58' 26''$, $B = 21° 9' 58''$, $c = 4151.7$.

16. $A = 84° 49' 58''$, $B = 28° 48' 26''$, $c = 42.374$.

17. $B = 24° 11' 24''$, $C = 144° 55' 48''$, $a = 186.98$.

18. $B = 20° 36' 34''$, $C = 102° 10' 14''$, $a = 37.5$.

19. 6.

20. 10.392.

21. $A = B = 90° - \frac{1}{2} C$,
$c = \dfrac{a \sin C}{\sin A}$.

22. 8.9212.

23. 25.

24. 3800 yd.

25. 729.67 yd.

26. 430.85 yd.

27. 10.266 mi.

28. 2.3385 and 5.0032.

Exercise 54. Page 125

1. $\frac{1}{2}[\log(s-b)+\log(s-c)+\operatorname{colog} s+\operatorname{colog}(s-a)]$.
2. $\frac{1}{2}[\log(s-b)+\log(s-c)+\operatorname{colog} b+\operatorname{colog} c]$.
3. $\frac{1}{2}[\log(s-a)+\log(s-b)+\log(s-c)+\operatorname{colog} s]$.
4. $\log r+\operatorname{colog}(s-a)$.
5. $\log(s-a)+\log\tan\frac{1}{2}A$.
6. The second.
7. $\sqrt{\frac{1}{7}}$, or 0.37796; 41° 24′ 34″.
9. $A=60°$.

Exercise 55. Page 127

1. 38° 52′ 48″; 126° 52′ 12″; 14° 15′.
2. 32° 10′ 55″; 136° 23′ 50″; 11° 25′ 15″.
3. 27° 20′ 32″; 143° 7′ 48″; 9° 31′ 40″.
4. 42° 6′ 13″; 56° 6′ 36″; 81° 47′ 11″.
5. 16° 25′ 36″; 30° 24′; 133° 10′ 24″.
6. 46° 49′ 35″; 57° 59′ 44″; 75° 10′ 41″.
7. 26° 29″; 43° 25′ 20″; 110° 34′ 11″.
8. 49° 34′ 58″; 58° 46′ 58″; 71° 38′ 4″.
9. 51° 53′ 12″; 59° 31′ 48″; 68° 35′.
10. 36° 52′ 12″; 53° 7′ 48″; 90°.
11. 36° 52′ 12″; 53° 7′ 48″; 90°.
12. 33° 33′ 27″; 33° 33′ 27″; 112° 53′ 6″.
13. 60°; 60°; 60°.
14. 28° 57′ 18″; 46° 34′ 6″; 104° 28′ 36″.
15. 36° 52′ 12″; 53° 7′ 48″; 90°.
16. 8° 19′ 9″; 33° 33′ 36″; 138° 7′ 15″.
17. 45°; 120°; 15°.
18. 45°; 60°; 75°.
19. 84° 14′ 34″.
20. 54° 48′ 54″.
21. 105°; 15°; 60°.
22. 54.516.
23. 60°.
24. 12.434 in.
25. 4° 23′ 2″ W. of N. or W. of S.
26. $A=90°\ 37'\ 3''$;
    $B=104°\ 28'\ 41''$;
    $C=96°\ 55'\ 44''$;
    $D=67°\ 58'\ 32''$.
27. 82° 49′ 10″.
28. 36° 52′ 11″;
    53° 7′ 49″.

Exercise 56. Page 128

1. 277.68.
2. 452.87.
3. 8.0824.
4. 27.891.
5. 139.53.
6. 1380.7.
7. 10,280.9.
8. 82,362.
9. 409.63.
10. 1,067,750.
12. 10.0067 sq. in.
13. 18.064 sq. in.
14. 13.41 sq. in.

Exercise 57. Page 129

1. 85.926.
2. 23,531.
3. 436,540.
4. 157.63.
5. 7,408,200.
6. 398,710.
7. 176,384.
8. 25,848.
9. 92.963.
10. 3176.7.
11. 5.729 sq. in.

Exercise 58. Page 131

1. 6.
2. 150.
3. 43.301.
4. 1.1367.
5. 10.279.
6. 16.307.
7. 1224.8 sq. rd.; 7.655 A.
8. 3.84.
9. 4.8599.
10. 101.4.
11. 62.354.
12. 0.19975.
13. 240.
14. 8160.
15. 26,208.
16. 17.3206.
17. 10.392.
18. 365.68.
19. 29,450; 6982.8.
20. 15,540.
21. 4,333,600.
22. 13,260.
24. 3 A. 0.392 sq. ch.
25. 12 A. 3.45 sq. ch.
26. 4 A. 6.634 sq. ch.
27. 61 A. 4.97 sq. ch.
28. 4 A. 6.633 sq. ch.
29. 13.93 ch., 23.21 ch., 32.50 ch.
30. 14 A. 5.54 sq. ch.
31. 30°; 30°; 120°.
32. 2,421,000 sq. ft.
33. 199 A. 8 sq. ch.
34. 210 A. 9.1 sq. ch.
35. 12 A. 9.78 sq. ch.
37. 876.34 sq. ft.
38. 1229.5 sq. ft.
39. 9 A. 0.055 sq. ch.
41. 1075.3.
42. 2660.4.
43. 16,281.
45. Area $=ab\sin A$.

Exercise 59. Page 133

1. 20 ft.
2. $37° 34' 5''$.
3. $30°$.
4. 199.70 ft.
5. 106.69 ft.; 142.85 ft.
6. 43.12 ft.
7. 78.36 ft.
8. 75 ft.
9. 1.4442 mi.
10. 56.649 ft.
11. 2159.5 ft.
12. 7912.8 mi.
13. 260.21 ft.; 3690.3 ft.
14. 2922.4 mi.
15. $60°$.
16. 3.2068.
17. 6.6031.
18. 238,410 mi.
19. 1.3438 mi.
20. 861,860 mi.
21. 235.81 yd.
22. $26° 34'$.
23. 69.282 ft.
24. $49° 18' 42''$; $40° 41' 18''$.
25. $50° 29' 35''$; $39° 30' 25''$.
26. $74° 44' 14''$.
27. 350.61 in.
28. 115.83 in.
29. 388.62 in.
30. $83° 37' 40''$.
31. $97° 11'$.
32. $89° 50' 18''$.
33. 0.2402; 1.9216 in.; 33.306 in.
34. 1.7 in.; 0.588 in.
35. $\dfrac{a-b}{a+b}$.
36. $30°$.
37. 97.86 in.; 153.3 in.; 159.31 in.
38. 1302.5 ft.; $33° 6' 51''$.
39. 0.9428.
41. 45 ft.
43. 0.9524.
44. $\dfrac{2h\sqrt{l^2+w^2}}{h^2-l^2-w^2}$.

Exercise 60. Page 137

4. 460.46 ft.
5. 88.936 ft.
6. 56.564 ft.
7. 51.595 ft.
8. 422.11 yd.
9. 41.411 ft.
10. 234.51 ft.
11. 12,492.6 ft.
12. 255.78 ft.
13. 529.49 ft.
14. 294.69 ft.
15. 101.892 ft.
16. 210.44 ft.
18. 19.8; 35.7; 44.5.
19. 13.657 mi. per hour.
20. N. $76° 56'$ E.; 13.938 mi. per hour.
21. 3121.1 ft.; 3633.5 ft.
22. 25.433 mi.
23. 6.3397 mi.
24. $x = \dfrac{OB \sin O}{\sin a}$; $\sin a$; $90°$; $B = 90°$; $\angle a = 90° - O$.
25. 288.67 ft.
26. 11.314 mi. per hour.
28. 658.36 lb.; $22° 23' 47''$ with first force.
29. 88.326 lb.; $45° 37' 16''$ with known force.
30. 757.50 ft.
31. 520.01 yd.
32. 1366.4 ft.
35. 536.28 ft.; 500.16 ft.
36. 345.46 yd.
37. 61.23 ft.

Exercise 61. Page 141

1. 19,647 sq. ft.
2. 27.527 sq. in.
3. 41.569 sq. in.
4. 6.
6. $\frac{1}{2}$; $\frac{1}{2}\sqrt{2}$.
9. 6.
11. 40,322.5 sq. ft.

Exercise 62. Page 142

1. 11.124 A.
2. 21.617 A.
3. 15.129 A.
4. 14 A.
5. 13.51 A.
6. 13.453 A.
7. 10 A.
8. 4.5348 A.; 10.4652 A.
9. 36.38 A.
10. 20.07 A.
11. 3.766 A.
12. 2.485 A.

Exercise 63. Page 144

1. 6.5223 sq. in.
2. 66.2343 sq. in.
3. 3.583 sq. in.; 27.6565 sq. in.
4. 8.6965 sq. in.
5. 112.26 sq. in.; 201.9 sq. in.
6. 0.14279.
7. 116.012 sq. in.
8. $\frac{3}{4}$.

Exercise 64. Page 147

1. $18' 23''$; 18.385 mi.
2. $37' 29''$; 37.4775 mi.
3. $51' 33''$; 34.445 mi.
4. $37' 16''$; 7.4135 mi.
5. $13' 53''$; 20.787 mi.
6. $19' 52''$; $12° 57' 8''$ S.
7. 35.207 mi.
8. 16.6296 mi.; $11' 6.7''$.
9. 59.155 mi.
10. 101.44 mi.
11. 11.483 mi.
12. 44.5 mi.
13. S. $75° 31' 20''$ E.; 23.2374 mi.
14. N. $17° 6' 14''$ W.; $32° 50' 30''$ N.
15. 23.854 mi.; S. $56° 58' 34''$ E.
16. 27.803 mi.; N. $52° 18' 21''$ W.

Exercise 65. Page 148

1. $66° 54' 39''$ W.
2. 103.57 mi.
3. $63° 9' 50''$ W.

Exercise 66. Page 149

1. $31° 26' 15''$ N.; $41° 44' 23''$ W.
2. S. $63° 26'$ W.; 42.486 mi.; $16° 14' 52''$ W.
3. $41° 50' 5''$ N.; $58° 15' 1''$ W.
4. 16.727 mi.; $30° 16' 19''$ W.
5. N. $77° 9' 38''$ W.; $32° 28' 32''$ W.
6. $40° 4' 16''$ N.; $72° 44' 56''$ W.
7. $42° 47' 43''$ N.; $70° 48' 25''$ W.

Exercise 67. Page 150

1. $35° 49' 10''$ S.; $22° 2' 44''$ W.; N. $61° 42'$ W.; 183.16 mi.
2. $42° 15' 29''$ N.; $69° 5' 11''$ W.; 44.939 mi.
3. $32° 53' 34''$ S.; $13° 1' 53''$ E; 287.16 mi.
4. $41° 1' 40''$ N.; $69° 54' 1''$ W.
5. $57' 19''$; 21.4 mi.
6. $1° 37' 8''$; 45.652 mi.

Exercise 68. Page 152

1. $\frac{3}{2}\pi$.
2. $\frac{1}{16}\pi$.
3. $\frac{5}{16}\pi$.
4. $\frac{1}{24}\pi$.
5. $\frac{131}{120}\pi$.
6. 8π.
7. $\frac{10}{9}\pi$.
8. $\frac{50}{3}\pi$.
9. 270°.
10. 240°.
11. 210°.
12. 225°.
13. $7° 30'$.
14. 540°.
15. 1080°.
16. 1800°.
17. II.
18. II.
19. III.
20. IV.
21. II.
22. II.
23. I.
24. III.
25. 216°, $\frac{6}{5}\pi$.
26. 300°, $\frac{5}{3}\pi$.
27. 120°, $\frac{2}{3}\pi$.
28. $33° 45'$, $\frac{3}{16}\pi$.
29. 0.017453; 0.0002909.
30. $3437.75'$; $206,265''$.
31. $\frac{1}{3}\pi$ radians.
32. $\frac{1}{2}\pi$ radians.

Exercise 69. Page 154

1. 16°, 164°, 376°, 524°.
2. 30°, 150°, 390°, 510°, 750°, 870°.
3. 30°, 150°, 390°, 510°, 750°, 870°, 1110°, 1230°.
4. $67° 30'$, $112° 30'$, $427° 30'$, $472° 30'$.
5. 18°, 162°, 378°, 522°.
6. 0.99999995769.
7. 0.00029088820.
8. 0.00029088821.
9. 0.00058177632.
10. 0.000582.
11. 0.0175.

Exercise 70. Page 155

1. 60°, 300°.
2. − 60°, − 300°.
3. 25°, 335°, 385°, 695°.
4. 60°, 300°, 420°, 660°.
5. 45°, 225°.
6. − 135°, − 315°.
7. 60°, 240°, 420°, 600°.
8. 30°, 210°, 390°, 570°.
9. 26° 34′, 206° 34′, 386° 34′, 566° 34′.
10. − 116° 34′, − 296° 34′, − 476° 34′, − 656° 34′.

Exercise 71. Page 156

5. 60°, 120°.
6. 45°, 135°.
7. 30°, 210°.
8. 90°, 270°.
9. 60°, 300°.
10. 135°, 225°.
11. $\frac{1}{2}\sqrt{3}$.
12. $\frac{1}{2}$.
13. $\frac{1}{2}\sqrt{2}$.
14. $\frac{1}{2}\sqrt{2}$.
19. 60°, 240°, 420°, 600°.
20. 58°, 238°, 418°, 598°.
21. 74°, 106°, 434°, 466°.
22. 19°, 161°, 379°, 521°.
23. 15° 24′ 30″, 195° 24′ 30″, 375° 24′ 30″, 555° 24′ 30″.
24. 19°, 341°, 379°, 701°.
25. 19° 28′ 17″, 160° 31′ 43″.
26. $\pm \frac{1}{2}\sqrt{2}$.
27. $\pm \frac{1}{2}\sqrt{3}$ or 0.

Exercise 74. Page 161

2. 360° or 2π.
4. 180° or π.
6. 180° or π.
8. 360° or 2π.
9. 180° and 360°.
10. Complements.

Exercise 75. Page 162

1. 270.63.
2. 416.65.
3. 2695.8.
4. 4.163.
5. Impossible.
6. Impossible.
7. 345.48 ft.
9. 40′ 9″.
10. − 175°, 185°, 535°, 545°.
11. − 200°, 160°, 560°, 520°.
12. 2 radians; 114° 35′ 30″.
13. $\frac{1}{3}$ radian; 19° 5′ 55″.
22. 30°, 210°, 390°, 570°.
23. 60°, 240°, 420°, 600°.

Exercise 77. Page 166

1. $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$.
2. 90° or 270°.
3. 21° 28′ or 158° 32′.
4. 0° or 90°.
5. 30°, 150°, 199° 28′, or 340° 32′.
6. 51° 19′, 180°, or 308° 41′.
7. 30°, 150°, or 270°.
8. 35° 16′, 144° 44′, 215° 16′, or 324° 44′.
9. 75° 58′ or 255° 58′.
10. 60°, 180°, or 300°.
11. 90° or 143° 8′.
12. 30°, 150°, 210°, or 330°.
13. 0°, 120°, 180°, or 240°.
14. 45°, 161° 34′, 225°, or 341° 34′.
15. 60°, 120°, 240°, or 300°.
16. 26° 34′ or 206° 34′.
17. 30° or 150°.
18. 45° or 135°.
19. 60°, 90°, 270°, or 300°.
20. 60°, 90°, 120°, 240°, 270°, or 300°.
21. 32° 46′, 147° 14′, 212° 46′, or 327° 14′.
22. $\tan^{-1}\frac{a^2-1}{2a}$.
23. $\cos^{-1}\left(\frac{-a \pm \sqrt{a^2+8a+8}}{4}\right)$.
24. 1.
25. 1.
26. 0°, 45°, 90°, 180°, 225°, or 270°.
27. 30°, 90°, 150°, 210°, 270°, or 330°.

28. 30°, 60°, 120°, 150°, 210°, 240°, 300°, or 330°.
29. 0°, 65° 42′, 180°, or 204° 18′.
30. 14° 29′, 30°, 150°, or 165° 31′.
31. 0°, 20°, 100°, 140°, 180°, 220°, 260°, or 340°.
32. 45°, 90°, 135°, 225°, 270°, or 315°.
33. 30°, 150°, or 270°.
34. 26° 34′, 90°, 206° 34′, or 270°.
35. 45°, 135°, 225°, or 315°.
36. 45°, 135°, 225°, or 315°.
37. 15°, 75°, 135°, 195°, 255°, or 315°.
38. 45°, 135°, 225°, or 315°.
39. 0°, 45°, 180°, or 225°.
40. 0°, 90°, 120°, 240°, or 270°.
41. 0°, 36°, 72°, 108°, 144°, 180°, 216°, 252°, 288°, or 324°.
42. 120°.
43. 54° 44′, 125° 16′, 234° 44′, 305° 16′.
44. 30°, 60°, 90°, 120°, 150°, 210°, 240°, 270°, 300°, or 330°.
45. $\sin^{-1} \pm \sqrt{\dfrac{k-1}{2}}$.
46. 90°, 216° 52′, or 323° 8′.
47. 30°, 90°, 150°, 210°, 270°, or 330°.
48. 0°, 45°, 180°, or 225°.
49. 45°, 60°, 120°, 135°, 225°, 240°, 300°, or 315°.
50. 0°, 45°, 135°, 225°, or 315°.
51. 90° or 270°.
52. $\frac{1}{2}\sqrt{3}$.
53. $\frac{1}{2}$.
54. 0°, 45°, 90°, 180°, 225°, or 270°.
55. 30°, 150°, 210°, or 330°.
56. 60°.
57. 105° or 345°.
58. 135°, 315°, or $\frac{1}{2}\sin^{-1}(1-a)$.
59. 30°, 60°, 120°, 150°, 210°, 240°, 300°, or 330°.
60. 60°, 90°, 120°, 240°, 270°, or 300°.
61. 0°, 90°, 180°, or 270°.
62. 0°, 90°, 120°, 180°, 240°, or 270°.
63. 0°, 74° 5′, 127° 25′, 180°, 232° 35′, or 285° 55′.
64. 0°, 180°, 220° 39′, or 319° 21′.
65. 8° or 168°.
66. 40° 12′, 139° 48′, 220° 12′, or 319° 48′.
67. 0°, 60°, 120°, 180°, 240°, or 300°.
68. 30° or 330°.
69. 60°, 120°, 240°, or 300°.
70. 18°, 90°, 162°, 234°, 270°, or 306°.
71. 30°, 60°, 120°, 150°, 210°, 240°, 300°, or 330°.
72. 53° 8′, 126° 52′, 233° 8′, or 306° 52′.
73. 30°.
74. 22° 37′ or 143° 8′.
75. 0°, 20°, 40°, 60°, 80°, 100°, 120°, 140°, 160°, 180°, 200°, 220°, 240°, 260°, 280°, 300°, 320°, or 340°.
76. $22\frac{1}{2}$°, 45°, $67\frac{1}{2}$°, 90°, $112\frac{1}{2}$°, 135°, $157\frac{1}{2}$°, $202\frac{1}{2}$°, 225°, $247\frac{1}{2}$°, 270°, $292\frac{1}{2}$°, 315°, or $337\frac{1}{2}$°.
77. 45° or 225°.
78. ± 1 or $\pm \frac{1}{7}\sqrt{21}$.
79. $\frac{1}{3}\sqrt{3}$ or $-\frac{1}{2}\sqrt{3}$.
80. 0 or ± 1.
81. 0°, 30°, 90°, 150°, 180°, 210°, 270°, or 330°.
82. 120° or 240°.
83. 60°, 120°, 240°, or 300°.
84. 10° 12′, 34° 48′, 190° 12′, or 214° 48′.
85. 29° 19′, 105° 41′, 209° 19′, or 285° 41′.
86. 0°, 45°, 90°, 180°, 225°, or 270°.
87. 0°, 45°, 135°, 225°, or 315°.
88. 0°, 60°, 120°, 180°, 240°, or 300°.
89. 27° 58′, 135°, 242° 2′, or 315°.

Exercise 78. Page 170

1. $x = a,\ y = 0;$ or $x = 0,\ y = a$.

2. $x = \sin^{-1} \pm \sqrt{\dfrac{a-b}{2}}$,
$y = \sin^{-1} \pm \sqrt{\dfrac{a+b}{2}}$.

3. $x = 76°\,10′,\ y = 15°\,30′$.

4. $x = 100,\ y = 200$.

5. $x = \sin^{-1} \pm \sqrt{\dfrac{m-n+1}{2}}$,
$y = \dfrac{m+n-1}{2}$.

6. $x = 90°$,
$y = 0°$ or 180°.

7. $x = \cos^{-1} \frac{1}{2}\left(a \pm \sqrt{b - a^2 + 2}\right)$; $y = \cos^{-1} \frac{1}{2}\left(a \pm \sqrt{b - a^2 + 2}\right)$.

8. $x = \tan^{-1} \frac{m}{n} \tan a + \frac{1}{2} \cos^{-1}[2m^2 - (2m^2 - 2n^2)\cos^2 a - 1]$;

$y = \tan^{-1} \frac{m}{n} \tan a - \frac{1}{2} \cos^{-1}[2m^2 - (2m^2 - 2n^2)\cos^2 a - 1]$.

9. $x = \tan^{-1} \frac{a}{b} + \cos^{-1} \frac{1}{2}\sqrt{a^2 + b^2}$; $y = \tan^{-1} \frac{a}{b} - \cos^{-1} \frac{1}{2}\sqrt{a^2 + b^2}$.

10. $x = 24° 13'$, $r = 225.12$; $x = 204° 13'$, $r = -225.12$.

11. $x = 42° 28'$, $r = 151$; $x = 222° 28'$, $r = -151$.

Exercise 79. Page 171

1. $\phi = 30°$ or $150°$; $x = 0.134$ or 1.866.
2. $\theta = \sin^{-1}(a - 1)$; $x = 2 - a$.
3. $\lambda = 45°, 135°, 225°,$ or $315°$; $\mu = 30°, 150°, 210°,$ or $330°$.
4. $\theta = \frac{1}{2}\sin^{-1}\left(\frac{a^2 + b^2}{2} - 1\right) + \frac{1}{2}\sin^{-1}\frac{a^2 - b^2}{a^2 + b^2}$;

$\phi = \frac{1}{2}\sin^{-1}\left(\frac{a^2 + b^2}{2} - 1\right) - \frac{1}{2}\sin^{-1}\frac{a^2 - b^2}{a^2 + b^2}$.
5. $\theta = \cos^{-1}\left[\pm\sqrt[4]{\frac{b^2}{a(b-a)}}\right]$; $\phi = \cos^{-1}\left[\pm\sqrt[4]{\frac{a}{b-a}}\right]$.
6. $\theta = 0°$.

Exercise 80. Page 172

1. $a^2 + b^2 - 2(a - b) = -1$.
2. $ab = 1$.
3. $(n - m)^2 + (q - p)^2 = 1$.
4. $b - a = \frac{1}{p}\sqrt{p^2 + q^2}$.
5. $bc = 1$.
6. $x = \pm\sqrt{2ry - y^2} + r \operatorname{versin}^{-1}\frac{y}{r}$.
7. $(m^2 + n^2 - 1)^2 = (n + 1)^2 + m^2$.
8. $a^{\frac{4}{3}}b^{\frac{2}{3}} + a^{\frac{2}{3}}b^{\frac{4}{3}} = 1$.
9. $(m + n)\sqrt{4 - (m - n)^2} = 2(m - n)$.
10. $p'r = -r'p$.
11. $k^4 + l^4 = 2kl(kl - 2)$.
12. $a^2b^2r^2 + a^2c^2q^2 + b^2c^2p^2 = a^2b^2c^2$.

Exercise 81. Page 176

1. 1; -1.
2. 1; $\sqrt{-1}$; -1; $-\sqrt{-1}$.
3. 1; $0.7660 + 0.6428\,i$; $0.1736 + 0.9848\,i$.
4. 1; $\frac{1}{4}\left(\sqrt{5} - 1 + i\sqrt{10 + 2\sqrt{5}}\right)$; $\frac{1}{4}\left(-\sqrt{5} - 1 + i\sqrt{10 - 2\sqrt{5}}\right)$;

$\frac{1}{4}\left(-\sqrt{5} - 1 - i\sqrt{10 - 2\sqrt{5}}\right)$; $\frac{1}{4}\left(\sqrt{5} - 1 - i\sqrt{10 + 2\sqrt{5}}\right)$.
5. 1; $\frac{1}{2} + \frac{1}{2}\sqrt{-3}$; $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$; -1; $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$; $\frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

$\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1}$; $\sqrt{-1}$; $-\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1}$; $-\frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{-1}$; $-\sqrt{-1}$;

$\frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{-1}$.
6. $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{-2}$; $-\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{-2}$; $-\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{-2}$; $\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{-2}$.

Exercise 82. Page 177

1. $-\frac{5}{2}+\frac{5}{2}\sqrt{-3}$; $-\frac{5}{2}-\frac{5}{2}\sqrt{-3}$; 5.
2. $\frac{3}{2}\sqrt{2}+\frac{3}{2}\sqrt{-2}$; $-\frac{3}{2}\sqrt{2}+\frac{3}{2}\sqrt{-2}$; $-\frac{3}{2}\sqrt{2}-\frac{3}{2}\sqrt{-2}$; $\frac{3}{2}\sqrt{2}-\frac{3}{2}\sqrt{-2}$.
3. $\frac{3}{2}+\frac{3}{2}\sqrt{-3}$; $-\frac{3}{2}+\frac{3}{2}\sqrt{-3}$; -3.
4. $2(\cos 36^\circ + i\sin 36^\circ)$; $2(\cos 72^\circ + i\sin 72^\circ)$; $2(\cos 108^\circ + i\sin 108^\circ)$.
5. $0.9980 + 0.0628\,i$; $0.9921 + 0.1253\,i$; $0.9823 + 0.1874\,i$.

Exercise 83. Page 183

7. 120.
8. 5040.
9. 720.
10. 40,320.
11. 3,628,800.
12. 604,800.
13. 90.
14. 42.
15. 15.
16. 6840.
17. 7.38883.
18. 1.64871.
19. $\cos 28^\circ 39'$.
20. $\cos 7^\circ 10'$.
21. $\cos 114^\circ 25' 32''$.
22. $\cos 0^\circ$.
23. $\sin 57^\circ 17' 48''$.
24. $\sin 28^\circ 38' 40''$.
25. $\sin 65^\circ 24' 45''$ or $\sin 114^\circ 35' 15''$.
26. $\sin 0^\circ$ or $\sin 180^\circ$.
27. $\tan 0^\circ$.
28. $\tan 56^\circ 40' 12''$.
29. $\tan 28^\circ 38' 20''$.
30. $\tan 86^\circ 23' 16''$.
35. $0.6931 + 2\pi i$; $0.6931 + 4\pi i$.
36. $1.3862 + 2\pi i$; $1.3862 + 4\pi i$.
37. $0.3465 + 2\pi i$; $0.3465 + 4\pi i$.
38. $0.6931 + \pi i$; $0.6931 + 3\pi i$.
39. $1.609 + 2\pi i$; $1.609 + 4\pi i$; $1.609 + 6\pi i$.
40. $3.218 + 2\pi i$; $3.218 + 4\pi i$; $3.218 + 6\pi i$.
41. $4.827 + 2\pi i$; $4.827 + 4\pi i$; $4.827 + 6\pi i$.
42. $1.609 + \pi i$; $1.609 + 3\pi i$; $1.609 + 5\pi i$.
43. $4.605170 + 2\pi i$; $4.605170 + 4\pi i$.
44. $2.302585 + \pi i$; $2.302585 + 3\pi i$.
45. $6.907755 + 2\pi i$; $6.907755 + 4\pi i$.
46. $1.151292 + 2\pi i$; $1.151292 + 4\pi i$.

Exercise 84. Page 184

1. 362.8 ft.
2. 1445.67 ft.; 1704.7 ft.; 1622.5 ft.
4. $\dfrac{2m(n^2-1)+2n(m^2-1)}{(m^2-1)(n^2-1)-4mn}$.
5. 2.
12. $b\sin C$.
13. 794.73 ft.

SPHERICAL TRIGONOMETRY

Exercise 85. Page 195

4. Either a or $b = 90°$.
5. $A = 90°$; $B = b$.
6. $B = 90°$; $A = a$.
7. $a = 90°$; $B = b = 90°$.
8. $a = b = c = 0°$.
9. $a = c = 90°$; $B = b = 90°$.
10. $A = 90°$; $B = b$.
11. $c = 90°$; $b = B = 90°$.

Exercise 86. Page 197

11. 98°; 103°; 111°.
12. $95\frac{1}{2}°$; $98\frac{1}{4}°$; $107\frac{5}{6}°$.
13. 101° 30′; 91°; 78°.
14. 96° 20′; 131° 3′; 76° 17′.
15. 83° 22′ 20″; 97° 30′ 30″; 111° 13′.
16. 136° 30′ 23″; 81° 37′ 7″; 92° 23′ 21″.
17. 111° 17′ 21″; 86° 11′ 53″; 90° 21′ 46″.
18. 101° 12′ 31″; 73° 23′ 18″; 90°.
19. 68° 30′ 17″; 90°; 90°.
20. 90°; 135°; $112\frac{1}{2}°$.
21. 109.5°; 99.3°; 78.4°.
22. 139.28°; 90°; 52.17°.

Exercise 87. Page 199

1. $c = 56° 10′ 25″$; $A = 37° 0′ 18″$; $B = 67° 14′ 23″$.
2. $c = 67° 28′ 45″$; $A = 44° 5′ 43″$; $B = 69° 38′ 22″$.
3. $c = 77° 22′ 43″$; $A = 46° 26′ 12″$; $B = 77° 3′ 37″$.
4. $c = 83° 19′ 25″$; $A = 56° 35′ 4″$; $B = 80° 0′ 24″$.
5. $c = 88° 38′ 19″$; $A = 63° 1′ 54″$; $B = 87° 19′ 35″$.
6. $c = 91° 7′ 24″$; $A = 68° 1′ 39″$; $B = 92° 46′ 55″$.
7. $c = 92° 3′ 52″$; $A = 75° 8′ 22″$; $B = 97° 43′ 51″$.
8. $c = 91° 23′ 5″$; $A = 82° 7′ 12″$; $B = 99° 54′ 17″$.
9. $c = 87° 30′ 8″$; $A = 94° 19′ 58″$; $B = 119° 54′ 19″$.
10. $c = 75° 58′ 18″$; $A = 116° 47′ 32″$; $B = 115° 38′ 35″$.
11. $c = 54° 20′$; $A = 46° 59′ 43″$; $B = 57° 59′ 19″$.
12. $c = 87° 11′ 40″$; $A = 88° 11′ 58″$; $B = 32° 42′ 39″$.
13. $c = 59° 4′ 26″$; $A = 63° 15′ 13″$; $B = 44° 26′ 22″$.
14. $c = 63° 55′ 43″$; $A = 105° 44′ 21″$; $B = 147° 19′ 47″$.
15. $c = 55° 9′ 33″$; $A = 27° 28′ 35″$; $B = 73° 27′ 10″$.
16. $c = 23° 50′$; $A = 37° 36′ 55″$; $B = 54° 49′ 20″$.
17. $c = 44° 33′ 18″$; $A = 49° 20′ 16″$; $B = 50° 19′ 24″$.
18. $c = 97° 13′ 4″$; $A = 131° 43′ 48″$; $B = 81° 58′ 54″$.
19. $c = 2° 3′ 56″$; $A = 77° 20′ 34″$; $B = 12° 39′ 55″$.
20. $c = 25° 14′ 40″$; $A = 54° 35′ 18″$; $B = 38° 10′ 9″$.
21. 54° 4′ 7″.
22. 71° 24′ 17″.
23. 143° 34′ 25″.
24. 54° 35′ 10″.
25. 7.9624 in.
26. 18.052 in.
27. $5598\frac{2}{189}$.
28. 63° 26′ 6″; 63° 26′ 6″
29. $5274\frac{37}{63}$ mi.

Exercise 88. Page 200

1. $b = 43° 32' 30''$; $A = 46° 59' 40''$; $B = 57° 59' 15''$.
2. $b = 32° 40' 8''$; $A = 88° 11' 58''$; $B = 32° 42' 53''$.
3. $b = 36° 54' 48''$; $A = 63° 15' 10''$; $B = 44° 26' 23''$.
4. $b = 150° 59' 43''$; $A = 105° 44' 15''$; $B = 147° 19' 45''$.
5. $b = 51° 53'$; $A = 27° 28' 38''$; $B = 73° 27' 11''$.
6. $b = 32° 41'$; $A = 49° 20' 16''$; $B = 50° 19' 16''$.
7. $b = 79° 13' 38''$; $A = 131° 43' 50''$; $B = 81° 58' 53''$.
8. $b = 56° 50' 51''$; $A = 54° 54' 40''$; $B = 63° 25' 2''$.
9. $b = 0° 27' 7''$; $A = 77° 20'$; $B = 12° 40' 40''$.
10. b is indeterminate; $A = 90°$; $B = b$.
11. $79° 7' 12''$.

Exercise 89. Page 201

1. $b = 36° 54' 49''$; $c = 59° 4' 26''$; $B = 44° 26' 18''$.
2. $b = 43° 32' 32''$; $c = 54° 19' 53''$; $B = 57° 59' 15''$.
3. $b = 32° 39' 54''$; $c = 87° 10'$; $B = 32° 42' 35''$.
4. $b = 150° 59' 44''$; $c = 63° 55' 40''$; $B = 147° 19' 48''$.
5. $b = 129° 38' 18''$; $c = 74° 3' 45''$; $B = 126° 46' 54''$.
6. $b = 113° 16'$; $c = 77° 44' 40''$; $B = 109° 56'$.
7. $b = 51° 52' 48''$; $c = 55° 9' 33''$; $B = 73° 27' 15''$.
8. $b = 19° 17' 5''$; $c = 23° 49' 51''$; $B = 54° 49' 27''$.
9. $b = 32° 41'$; $c = 44° 33' 18''$; $B = 50° 19' 18''$.
10. Impossible.
11. $b = 28° 14' 31''$; $c = 78° 53' 20''$; $B = 28° 49' 57''$; or
    $b = 151° 45' 29''$; $c = 101° 6' 40''$; $B = 151° 10' 3''$.
12. $b = 79° 14'$; $c = 97° 13'$; $B = 81° 58' 30''$.

Exercise 90. Page 202

1. $b = 30° 8' 39''$; $c = 59° 51' 21''$; $A = 70° 17' 35''$.
2. $b = 49° 59' 58''$; $c = 91° 47' 40''$; $A = 92° 8' 23''$.
3. $b = 15° 16' 50''$; $c = 25° 14' 38''$; $A = 54° 35' 17''$.
4. $b = 56° 50' 49''$; $c = 69° 25' 13''$; $A = 54° 54' 40''$.
5. $b = 127° 4' 32''$; $c = 112° 47' 58''$; $A = 56° 11' 57''$.
6. $a = 92° 47' 33''$; $b = 50°$; $B = 50° 2'$.
7. $a = 20° 20' 23''$; $b = 15° 16' 52''$; $B = 38° 10' 7''$.
8. $a = 54° 30'$; $b = 30° 8' 35''$; $B = 35° 29' 56''$.
9. $a = 50°$; $b = 127° 4' 30''$; $B = 120° 3' 50''$.
10. $a = 2° 0' 55''$; $b = 0° 27' 10''$; $B = 12° 40'$.
12. $A = 175° 57' 10''$; $B = 135° 42' 50''$; $C = 135° 34' 7''$.
13. $a = 35° 47' 33''$; $A = 45° 33' 23''$; $B = 59° 40' 53''$.
14. $a = 61° 5' 43''$; $b = 29° 1' 56''$; $B = 32° 22' 32''$.
15. $b = 43° 13' 10''$; $c = 104° 25' 59''$; $A = 103° 59' 44''$.
16. $a = 66° 48' 12''$; $b = 29° 44' 10''$; $B = 31° 51' 34''$.
17. $a = 26° 3' 51''$; $A = 35°$; $B = 65° 46'$.
18. The triangle is impossible.
19. $a = 60° 16' 17''$; $b = 29° 41' 4''$; $B = 33° 16' 54''$.
20. $b = 42° 10' 17''$; $c = 106° 37' 37''$; $A = 105° 41' 30''$.

Exercise 91. Page 203

1. $a = 50° 0' 4''$; $b = 143° 5' 12''$; $c = 120° 55' 34''$.
2. $a = 120° 10' 3''$; $b = 119° 59' 46''$; $c = 75° 26' 58''$.
3. $a = 36° 27' 7''$; $b = 43° 32' 30''$; $c = 54° 20' 3''$.
4. $a = 90°$; $b = 88° 24' 35''$; $c = 90°$.
5. $a = 92° 47' 32''$; $b = 50°$; $c = 91° 47' 40''$.
6. $a = 1° 59' 30''$; $b = 0° 26' 48''$; $c = 2° 2' 28''$.
7. $a = 20° 20' 24''$; $b = 15° 17' 20''$; $c = 25° 14' 50''$.
8. $a = 54° 30'$; $b = 30° 8' 38''$; $c = 59° 51' 26''$.
9. $a = 50° 0' 4''$; $b = 56° 50' 51''$; $c = 69° 25' 11''$.
10. $a = 50°$; $b = 127° 4' 32''$; $c = 112° 48'$.

Exercise 92. Page 204

1. $A = 39° 29' 40''$; $B = C = 77° 0' 25''$; $b = 50°$.
2. $A = 46° 31' 22''$; $B = C = 77° 52' 10''$; $b = 60°$.
3. $A = 55° 52' 30''$; $B = C = 76° 17' 32''$; $b = 62° 37'$.
4. $A = 153° 45' 58''$; $C = 15°$; $a = 57° 28' 32''$; $b = 29° 35'$.
5. $A = 143° 18' 28''$; $C = 42° 30'$; $a = 124° 27' 44''$; $b = 68° 47'$.
6. $A = 156° 30' 56''$; $C = 49° 37'$; $a = 149° 0' 32''$; $b = 79° 49'$.
7. $\cos B = \cot b \tan \frac{1}{2} a$; $\sin \frac{1}{2} A = \csc b \sin \frac{1}{2} a$; $\cos AD = \cos b \sec \frac{1}{2} a$.

Exercise 93. Page 205

1. $\sin a \sin B = \sin b$; $\sin a \sin C = \sin c$.
2. $\sin a = \sin b \sin A$; $\sin b \sin C = \sin c$.
3. $\sin a = \sin c \sin A$; $\sin b = \sin c \sin B$.
4. $\sin B = \sin b \sin A$; $\sin C = \sin c \sin A$.
5. $\sin a = \sin b$; $\sin c = \sin b \sin C = \sin a \sin C$.
6. $\sin B = \sin A$; $\sin C = \sin c \sin B = \sin c \sin A$.
7. $\sin B = \sin b$; $\sin C = \sin c$.
8. $\sin C = \sin c$.
9. $\sin a = \sin b$; $\sin a = \sin c$; $\sin b = \sin c$.

Exercise 94. Page 206

1. $\cos a = \cos b \cos c$.
2. $\cos b = \cos a \cos c$.
3. $\cos a = \cos b \cos c$;
   $\cos b = \cos a \cos c$.
4. $\cos a = \cos b \cos c$;
   $\cos b = \cos a \cos c$;
   $\cos c = \cos a \cos b$.
7. $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$;
   $\cos B = \dfrac{\cos b - \cos c \cos a}{\sin c \sin a}$;
   $\cos C = \dfrac{\cos c - \cos a \cos b}{\sin a \sin b}$.

Exercise 95. Page 207

1. $1 + \cos B \cos C = \sin B \sin C \cos a$;
   $\cos B = -\cos C$;
   $\cos C = -\cos B$.
2. $\cos B \cos C - 1 = \sin B \sin C \cos a$;
   $\cos B = \cos C$;
   $\cos C = \cos B$.
3. $\cos B \cos C = \sin B \sin C \cos a$;
   $\cos B = \sin C \cos b$;
   $\cos C = \sin B \cos c$.
4. $\sin C \cos a = 0$;
   $\sin C \cos b = 0$;
   $\cos C = \cos c$.

8. $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$.

9. $\cos b = \dfrac{\cos B + \cos A \cos C}{\sin A \sin C}$.

10. $\cos c = \dfrac{\cos C + \cos A \cos B}{\sin A \sin B}$.

11. $\cos C = \dfrac{-\cos A + \sin B \sin C \cos a}{\cos B}$.

Exercise 96. Page 209

3. 147° 56′ 58″. **4.** 130° 45′ 47″. **5.** 126° 33′ 32″. **6.** 106° 40′ 42″. **7.** 27° 43′ 30″. **8.** 22° 16′. **9.** 17° 36′. **10.** 22° 58′.

Exercise 97. Page 211

1. $\sin \frac{1}{2} a = \sqrt{\dfrac{-\cos S \cos (S-A)}{\sin C}}$.

2. $\sin \frac{1}{2} a = \sqrt{\dfrac{-\cos S \cos (S-A)}{\sin B}}$.

3. $\sin \frac{1}{2} a = \sqrt{-\cos S \cos (S-A)}$.

4. $\sin \frac{1}{2} b = \sqrt{\dfrac{-2 \cos S \cos (S-B)}{\sqrt{2} \sin C}}$.

5. $\sin \frac{1}{2} b = \sqrt{\dfrac{-2 \cos S \cos (S-B)}{\sqrt{2} \sin A}}$.

6. $\sin \frac{1}{2} b = \sqrt{-2 \cos S \cos (S-B)}$.

7. $\frac{1}{2} c = -(33° 40' 32'')$.

8. $c = 90°$.

14. $\tan \frac{1}{2} b = \sqrt{-\cos S \cos (S-B) \sec (S-A) \sec (S-C)}$.

15. $\cos \frac{1}{2} b = \sqrt{\cos (S-A) \cos (S-C) \csc A \csc C}$.

16. $\sin \frac{1}{2} c = \sqrt{-\cos S \cos (S-C) \csc A \csc B}$.

Exercise 98. Page 214

1. $A = 63° 15' 11''$; $B = 132° 17' 58''$; $c = 59° 4' 17''$.
2. $A = 129° 58' 2''$; $B = 63° 15' 8''$; $c = 55° 52' 40''$.
3. $B = 88' 12' 24''$; $C = 55° 52' 42''$; $a = 50° 1' 40''$.
4. $B = 56° 11' 57''$; $C = 123° 21' 12''$; $a = 67° 11' 47''$.
5. 88° 57′ 50″.

Exercise 99. Page 215

1. 59° 3′. **2.** 54° 17′ 23″. **3.** 56° 48′ 16″. **4.** 66° 9′ 50

Exercise 100. Page 216

1. $\tan \frac{1}{2}(b-c) = \dfrac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} \tan \frac{1}{2} a$; $\cos \frac{1}{2} A = \dfrac{\sin \frac{1}{2}(B+C)}{\cos \frac{1}{2}(b-c)} \cos \frac{1}{2} a$.

2. $\tan \frac{1}{2}(a-c) = \dfrac{\sin \frac{1}{2}(A-C)}{\sin \frac{1}{2}(A+C)} \tan \frac{1}{2} b$; $\cos \frac{1}{2} B = \dfrac{\sin \frac{1}{2}(A+C)}{\cos \frac{1}{2}(a-c)} \cos \frac{1}{2} b$.

3. $\tan \frac{1}{2}(b-c) = \dfrac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} \tan \frac{1}{2} a$; $\tan \frac{1}{2}(b+c) = \dfrac{\cos \frac{1}{2}(B-C)}{\cos \frac{1}{2}(B+C)} \tan \frac{1}{2} a$.

4. $a = 39° 35' 51''$; $b = 60° 46' 23''$; $C = 132° 33' 38''$.
5. $a = 34° 20' 42''$; $b = 54° 37' 52''$; $C = 107° 11' 4''$.
6. $a = 46° 51' 6''$; $b = 61° 26' 40''$; $C = 103° 50' 16''$.
7. $a = 78° 7' 34''$; $b = 30° 26' 8''$; $C = 100° 29' 20''$.
8. $a = 36° 3' 9''$; $b = 36° 3' 9''$; $C = 71° 3' 46''$.
9. $a = 78° 18' 28''$; $b = 78° 18' 28''$; $C = 82° 3' 16''$.
10. $a = 36° 31' 44''$; $b = 121° 17' 44''$; $C = 161° 9' 52''$.

11. $a = 125° 8' 46''$; $b = 82° 53' 36''$; $C = 126° 58' 10''$.
12. $a = 152° 21' 47''$; $c = 88° 1' 39''$; $B = 77° 31'$.
13. $a = 127° 38' 22''$; $c = 106° 48' 22''$; $B = 54° 36'$.
14. $a = 100° 30' 12''$; $c = 93° 13' 46''$; $B = 72° 16' 59''$.
15. $a = 120° 27' 21''$; $c = 95° 51' 43''$; $B = 76° 1' 36''$.

Exercise 101. Page 217

1. $145° 49' 7''$.
2. $147° 7' 53''$.
3. $129° 25' 22''$.
4. $99° 4' 55''$.
5. $161° 22' 15''$.
6. $127° 22' 4''$.

Exercise 102. Page 219

1. $104° 16' 15''$.
2. $113° 32' 20''$.
3. $120° 21' 37''$.
4. $c = 120° 57' 27''$; $B = 116° 42' 30''$; $C = 116° 47'$.
5. $c_1 = 55° 42' 8''$; $B_1 = 120° 47' 45''$; $C_1 = 97° 42' 55''$;
   $c_2 = 23° 57' 17''$; $B_2 = 59° 12' 15''$; $C_2 = 29° 8' 39''$.
6. $c = 45° 12' 19''$; $B = 90°$; $C = 45° 44' 5''$.
7. Impossible.
8. $C = 51° 16' 40''$.
9. $A = 77° 21' 12''$.
10. $A = 78° 17' 48''$.
11. $B = 61° 34' 46''$.
12. $B = 72° 42'$.
13. $C = 146° 37' 42''$.
14. $C = 136° 24' 8''$.
15. $C = 105° 59' 24''$.
16. $b = 152° 43' 51''$; $c = 88° 12' 21''$; $A = 78° 15' 48''$.
17. $a = 128° 41' 46''$; $c = 107° 33' 20''$; $B = 55° 47' 40''$.

Exercise 103. Page 220

1. $b = 155° 56' 46''$; $c = 29° 2' 32''$; $C = 65° 51' 56''$.
2. No solution.
3. No solution.
4. $b = 100° 32'$; $c = 55° 55' 40''$; $C = 56° 54' 52''$.
5. $a = 149° 57' 12''$; $c = 106° 8' 15''$; $A = 149° 46' 12''$.
6. $a = 115° 23' 30''$; $b = 82° 30' 48''$; $B = 84° 4' 28''$.
7. $155° 5' 18''$.
8. $123° 3' 29''$.
9. $147° 41' 50''$.
10. The triangle is impossible.

Exercise 104. Page 221

1. $A = 113° 50' 38''$; $B = 66° 9' 22''$; $C = 97° 2' 52''$.
2. $A = 57° 41' 8''$; $B = 90° 55' 22''$; $C = 122° 18' 56''$.
3. $A = 130° 54' 22''$; $B = 112° 0' 38''$; $C = 100° 37' 24''$.
4. $A = 19° 10' 4''$; $B = 56° 14' 22''$; $C = 115° 34'$.
5. The triangle is impossible.
6. $A = 54° 1' 2''$; $B = 76° 36' 50''$; $C = 125° 58' 58''$.
7. $116° 44' 50''$.
8. $59° 4' 28''$.
9. $132° 14' 22''$.
10. $20° 9' 56''$.

Exercise 105. Page 222

1. $a = 125° 13' 2''$; $b = 118° 59' 44''$; $c = 70° 0' 48''$.
2. $a = 46° 31' 22''$; $b = 55° 36' 28''$; $c = 46° 31' 22''$.
3. $a = 103° 41'$; $b = 53° 55' 6''$; $c = 99° 35' 50''$.
4. The triangle is impossible.
5. $a = b = c = 69° 33' 42''$.
6. $a = 95° 22' 58''$; $b = 102° 26' 46''$; $c = 108° 11' 56''$.

7. $a = 139° 21' 22''$; $b = 126° 57' 52''$.
8. $a = 51° 17' 32''$; $b = 64° 2' 48''$.
9. $a = 104° 25' 10''$; $b = 53° 49' 26''$.
10. $a = 31° 9' 14''$; $b = 84° 18' 28''$.
11. The triangle is impossible.
12. $a = 99° 5' 46''$; $b = 42° 11' 54''$.
13. $a = 42° 20' 44''$; $b = 154° 37' 50''$.
14. $a = 121° 59' 28''$; $b = 112° 32' 44''$.

Exercise 106. Page 223

1. 8.7265.
2. 3.2724.
3. 50.729.
4. 1505.8.

Exercise 107. Page 225

1. 5°.
2. 80°.
3. 120°.
4. 103°.
5. 100°.
6. 111°.
7. $1.2682\,r^2$.
8. $1.9145\,r^2$.
9. $2.1418\,r^2$.
10. $1.3843\,r^2$ or $0.12595\,r^2$.
11. $0.87042\,r^2$.
12. $1.9635\,r^2$.
13. $1.2164\,r^2$.
14. $0.72372\,r^2$.

Exercise 108. Page 226

1. $7° 15' 59''$.
2. $216° 40' 20''$.
3. $133° 48' 53''$.
4. $2.2298\,r^2$.
5. $1.4956\,r^2$ or $0.17085\,r^2$.
6. $0.95484\,r^2$.
7. $0.024832\,r^2$.
8. $1.1891\,r^2$.
9. $0.7105\,r^2$.
10. $0.09301\,r^2$.
11. $2.8624\,r^2$.
12. $3.1416\,r^2$.
13. $5.4206\,r^2$.
14. 2070.1 sq. mi.
15. $\sin \frac{1}{2} A = \frac{1}{2} \sec \frac{1}{2} a$.
16. $\sin \frac{1}{2} A = \sec \frac{1}{2} a \cos \dfrac{180°}{n}$;
$\sin R = \sin \frac{1}{2} a \csc \dfrac{180°}{n}$;
$\sin r = \tan \frac{1}{2} a \cot \dfrac{180°}{n}$.
17. Tetrahedron, $70° 31' 46''$;
hexahedron, 90°;
octahedron, $109° 28' 14''$;
dodecahedron, $116° 33' 45''$;
icosahedron, $138° 11' 36''$.
18. $14° 19'$.

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